Lecture 1

Definitions:

	Linear equation in n unknowns	$m \times n$ system:
Definition		lin. system of m equations in n unknowns.
Regular form:	$a_1x_1 + a_nx_n = b$	$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$ $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$ $\vdots$ $a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$
Solution:		n numbers, satisfies all m equations
Example		2×2: $7x_1 + x_2 = 2$, $x_1 + 4x_2 = 3$ 2×3: $x_1 + x_2 - x_3 = 2$, $2x_1 + x_2 + x_3 = 4$ 3×2: $x_1 + x_2 = 2$, $x_1 - x_2 = 1$, $x_1 = 4$

$a_1, \ldots, a_n \& b$ or a_{ij}, b_i real numbers, $x_1, \ldots, x_n$ in both situations variables.

TYPE OF SOLUTIONS:
(1) inconsistent: no solutions.
(2) consistent: if it has at least one solution.
(3) Solution set: set of all solutions

Example:

System	$x_1 + x_2 = 2$ $x_1 - x_2 = 2$	$x_1 + x_2 = 2$ $x_1 + x_2 = 1$	$x_1 + x_2 = 2$ $-x_1 - x_2 = -2$
Type of solution	Consistent Exactly one solution $(x_1, x_2) = (2, 0)$	incosistent	consistent infintely many solutions, $(x_1, x_2) = (a, 2-a)$ where $a \in \mathbb{R}$

Definitions (2):

EQUIVALENT SYSTEMS: 2 lin. system of equations, same number of unknowns, same solution set.

Consider the linear system of equations, like in the tabel above, then we have:

Coefficient matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right)$$

PIVOT: the first nonzero entry in the PIVOTAL ROW: The first row.

SQUARE MATRIX: when then number of rows are equal to the number of unknowns.

ELEMENTARY ROW OPERATION:
(1) Interchange two rows
(2) Multiply a row by a nonzero real number
(3) Replace a row by its sum with a multiple of another row.

Strict triangular form:
kth equation, coefficient of the first $k-1$ variable are all zero, coefficient of x_k is nonzero

Type of systems:
Overdetermined system:if there are more equations then unknowns. Usually (not always) inconsistent.
Underdetermined system: if there are fewer equations than unknowns.
Homogeneous: If the constants on the right-handside are all zero. Are always consistant. Has a nontrivial solution of $n > m$

Type of variables.
Lead variables: the first nonzero element in each row of the row echelon form.
Free variables: All elements that are not lead variables.

Row echelon form:

(1) first nonzero entry each nonzero row is 1.
(2) row k does not consist entirely zeros, number of leading zero's in $k+1$ greater than number leading zero's k
(3) Rows with entries all zero, below rows having nonzero entries.

Gaussian elimination: The proces of using row operations to transform a linear system into one whose augmented matrix is in row echelon form.

Examples:

Lead and free variables:
When we have the matrix:
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ then x_1, x_3, x_5 are the lead variables, and the other variables (x_2, x_4) are the free variables.

Row echelon form and not row echelon form:

Row echelon form	$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	
NOT Row echelon form:	$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Bring a augmented matrix in Row echelon form:
1:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right) \to \begin{array}{c} (3) = (3) - 2(2) \\ (4) = (4) - (2) \\ (5) = (5) - (2) \end{array} \to \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \to \begin{array}{c} (4) = (4) - (3) \\ (5) = (5) - (3) \end{array} \to \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right)$$

$$\to \begin{array}{c} (4) = -\frac{1}{2}(4) \\ (5) = (5) + 3(4) \end{array} \to \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

These last matrix is incosistent. We see that in the last two rows. They say namely that $0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 1$ so $0 = 1$ which is not true.

2:

$$\left(\begin{array}{ccccc|c} -1 & -4 & 0 & -5 & -1 & -6 \\ 5 & 20 & 1 & 29 & 5 & 34 \\ 4 & 16 & 1 & 24 & 1 & 30 \end{array}\right) \rightarrow (1) = -(1) \rightarrow \left(\begin{array}{ccccc|c} 1 & 4 & 0 & 5 & 1 & 6 \\ 5 & 20 & 1 & 29 & 5 & 34 \\ 4 & 16 & 1 & 24 & 1 & 30 \end{array}\right) \rightarrow \begin{array}{c} (2)=(2)\text{-}5(1) \\ (3)=(3)\text{-}4(1) \end{array} \rightarrow \left(\begin{array}{ccccc|c} 1 & 4 & 0 & 5 & 1 & 6 \\ 0 & 0 & 1 & 4 & 0 & 4 \\ 0 & 0 & 1 & 4 & -3 & 6 \end{array}\right)$$

$$\rightarrow (3) = (3) - (2) \rightarrow \left(\begin{array}{ccccc|c} 1 & 4 & 0 & 5 & 1 & 6 \\ 0 & 0 & 1 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & -3 & 2 \end{array}\right) \rightarrow (3) = -\tfrac{1}{3}(3) \rightarrow \left(\begin{array}{ccccc|c} 1 & 4 & 0 & 5 & 1 & 6 \\ 0 & 0 & 1 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -\frac{2}{3} \end{array}\right)$$

Lead variables: x_1, x_3, x_5
Free variables: x_2, x_4
By back sabstitution:
$x_5 = 2\&x_3 = 4 - 4x_4\&x_1 = 4 - 4x_2 - 5x_4$

Definition (3)

A matrix is in the REDUCED ROW ECHELON FOR when:
(1) It is in row echelon form
(2) The first nonzero entry in each ROW is the only nonzero in its COLUMN.

GAUSS-JORDAN REDUCTION: The process of using elementary row operations to transform a matrix into reduced row echelon form.

Example:

IN REDUCED ROW ECHELON FORM:	NOT IN THE REDUCED ROW ECHELON FORM
$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

FROM ROW ECHELON FORM TO REDUCED ROW ECHELON FORM:
The professor used other matrices/ But every matrix in row echelon form can be reduced to a matrix in reduced row echelon form, so i will use the matrix we already used before.

1:
For the first matrix, I used the matrix like before, but then with 3 equations to make it consistent. So I will use:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \rightarrow (1) = (1) - (2) \rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \rightarrow (1) = (1) + (3) \rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \rightarrow$$

$$(2) = (2) - 2(3) \rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$ which is in the reduced row echelon form.

2:

$$\left(\begin{array}{ccccc|c} 1 & 4 & 0 & 5 & 1 & 6 \\ 0 & 0 & 1 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -\frac{2}{3} \end{array}\right) \rightarrow (1) = (1) - (3) \rightarrow \left(\begin{array}{ccccc|c} 1 & 4 & 0 & 5 & 0 & 6\frac{2}{3} \\ 0 & 0 & 1 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -\frac{2}{3} \end{array}\right)$$ which is already in the reduced row echelon form.

Application:

Kirchoff's Laws:

(1) At every node the sum of the incoming currents equal the sum of the outgoing currents.
(2) Around ever closed loop, the algebraic sum of the voltage gains must equal the algebraic sum of the voltage drops.

The voltage drops E for each resistor by Ohm's Law: $E = iR$ where i current in amperes, R in ohms.

Chemical equations:

Photosynthesis:
$x_1CO_2 + x_2H_2O \to x_3O_2 + x_4C_6H_{12}O_6$
First balans carbon, oxigen and hydrogen:

H	O	C
$2x_2 = 12x_4$	$2x_1 + x_2 = 2x_3 + 6x_4$	$x_1 = 6x_4$

$x_1 - 6x_4 = 0$
$2x_1 + x_2 - 2x_3 - 6x_4 = 0$
$2x_2 - 12x_4 = 0$
So we see that $x_1 = x_2 = x_3 = 6x_4$
If we take $x_4 = 1$,then $x_1 = x_2 = x_3 = 6$
So then we have: $6CO_2 + 6H_2O \to 6O_2 + C_6H_{12}O_6$

Lecture 2

Theorem:

Every $m \times n$ homogeneous system has a nontrivial solution if $n > m$

Proof:

(-) homogeneous system $\Rightarrow$ at most m nonzero row's $\Rightarrow$ at most m lead variables.
(-) n unknowns but $n > m$ so at least $n - m > 0$ free variables.
(-) Choose at least one free variables nonzero $\rightarrow$ nonzero solution.

Notations for matrices and vectors:

Matrices denoted by capital letters.
When matrix A has i rows and j columns then $A = (a_{ij})$
$m \times n$ matrix
$\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ matrices with real enteries.

Row vector	Column Vector
$1 \times n$ matrix	$n \times 1$ matrix
$\vec{\mathbf{y}} = \begin{pmatrix} y_1 & y_n & \cdots & y_m \end{pmatrix}$	$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

So when A is an $m \times n$ matrix, then $A = \begin{pmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$

Definition (2)

Algebraic properties:

Let's say we have two $m \times n$ matrices: $A \& B$
(-) $A = B$ iff $a_{ij} = b_{ij}$ for each i and j
(-) αA is the $m \times n$ matrix whose entry is αa_{ij}
(-) $A = (a_{ij})$ and $B = (b_{ij})$ then $A + B = (a_{ij} + b_{ij})$ which is $m \times n$
(-)ZERO MATRIX O the matrix whose entries are all zero.
(-) $C = AB$ which is an m times m matrix. Only defined when the number of columns of A equal to the number of rows of B. Calculation by Falk's scheme.
So $AB = (A\mathbf{b}_1, \dots, A\mathbf{b}_n)$ and $c_{ij} = \vec{\mathbf{a}}_i \mathbf{b}_j = \sum_{k=1}^{b} a_{ik} b_{kj}$

Matrix multiplication and linear systems:

We see that $A\mathbf{x} = \mathbf{b}$ where:

$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

Because $A\mathbf{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} \vec{\mathbf{a}}_1\mathbf{x} \\ \vec{\mathbf{a}}_2\mathbf{x} \\ \vdots \\ \vec{\mathbf{a}}_n\mathbf{x} \end{pmatrix} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n$

Example:

Scaler multiplication: $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 5 \end{pmatrix} \Rightarrow 3A = \begin{pmatrix} 3 & 6 & 0 \\ 9 & 0 & 15 \end{pmatrix}$
Sum: $\begin{pmatrix} 1 & 0 \\ 2 & 5 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ 6 & 0 \\ 7 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 8 & 5 \\ 7 & 6 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$
Multiplication:
$A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \\ 5 & 4 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 & 4 \\ 2 & -2 & -4 \end{pmatrix}$
Calculate $C = AB$

	$\begin{pmatrix} 0 & 3 & 4 \\ 2 & -2 & -4 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 \\ 3 & 2 \\ 5 & 4 \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 5 & 4 \\ 8 & 7 & 4 \\ 6 & 0 & -4 \end{pmatrix}$

So $C = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 5 & 4 \\ 8 & 7 & 4 \\ 6 & 0 & -4 \end{pmatrix}$

Explanation first entry:
$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} \Rightarrow c_{11} = a_{11}b_{11} + a_{12}b_{21} = 1 \cdot 0 + 0 \cdot 2 = 0 + 2 = 2$
Linear combination of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$:
$c_1\mathbf{a}_1 + \ldots + c_n\mathbf{a}_n$ if $\mathbf{a}_1, \ldots, \mathbf{a}_n$ vectors in $\mathbb{R}^m$ and $c_1, \ldots, c_n$ scalers.

Consistency Theorem linear systems: linear system $A\mathbf{x} = \mathbf{b}$ consistent iff $\mathbf{b}$ can be written as linear combination of the column vectors of A

Example:

Is the linear system:
$x_1 + 2x_2 = 1$
$2x_1 + 4x_2 = 1$
Consistent? The answer is yes because:
The vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ can be written as a linear combination of the column vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.
$x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} x_1+2x_2 \\ 2x_1+4x_2 \end{pmatrix}$

Algebra of matrices and the transpose of a matrix:

Provided indicated operations are defined, the following statement holds for all scalers α, β and matrices A, B, C
$A + B = B + A$
$(A + B) + C = (A + B) + C$
$(AB)C = A(BC)$
$A(B + C) = (AB + (AC)$
$(A + B)C = (AC + BC)$
$(\alpha\beta)A = \alpha(\beta A)$
$\alpha(AB) = (\alpha A)B = A(\alpha B)$
$(\alpha + \beta)A = \alpha A + \beta A$
$\alpha(A + B) = \alpha A + \alpha B$

TRANSPOSE:
(-) Transpose of $m \times n$ matrix A is the $n \times m$ matrix $B = (b_{ij})$ where $b_{ij} = a_{ji}$
(-) A^T is the transpose of A
(-) SYMMETRIC MATRIX: square matrix where $A^T = A$ for example I_n
(-) Algebraic rules:
$(A^T)^T = A$
$(\alpha A)^t = \alpha A^T$
$(A+B)^T = A^t + B^T$
$(AB)^T = B^T A^T$

Examples:

A	$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$
A^T	$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$
			Symmetric

Identity matrix:

The $n \times n$ identity matrix is the matrix $I_n = (\delta_{ij})$ where $\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$

For example:
$I_1 = 1 \& I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Multiplicative inverse:

NONSIGNULAR OR INVERTIBLE MATRIX: if there exists an matrix $B_{n\times n}$ s.t. $AB = BA = I_n$ then A is nonsingular.
Notation inverse of nonsingular matrix A is A^{-1}
SINGULAR: when an $n \times n$ matrix does not hav ea multiplicative inverse. An $N \times n$ matrix is said to be nonsingular if it does not have a multiplicative inverse.

Theorem:

(1) a matrix can have at most one inverse.
(2)If A and B are nonsingular $n \times n$ matrix then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

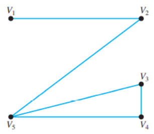
Proof:

(1) Suppose B and C both inverses of A,where A is nonsingular
$B = BI = B(AC) = (BA)C = IC = C$
So $B = C$ so A has one multiplicative inverse.
(2) $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$
$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1}AA^{-1} = I_n$
$\rightarrow AB$ is nonsignular and its inverse is $B^{-1}A^{-1}$

Examples:

Find the multiplicative inverse of $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
Define $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$
So we see that $AB = \begin{pmatrix} b_{11}+b_{21} & b_{12}+b_{22} \\ 0 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} b_{11} & b_{11} \\ b_{21} & b_{21} \end{pmatrix}$
we want that those matrices both equal $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
So: $\begin{pmatrix} b_{11}+b_{21} & b_{12}+b_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{11} \\ b_{21} & b_{21} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
We see that $b_{11} + b_{12} = b_{11} = 1$ therefore $b_{11} = 0$
We also see that $b_{12} + b_{21} = b_{11}$ we said that $b_{11} = 0$, but $b_{12} + b_{21} = b_{11}$ must equal 1
Therefore, A has no multiplicative inverse.

Application:



This is a walk on a graph, which we also have seen at Kaleidoscope (part 2).
We define a_{ij} as follows:

$$a_{ij} = \begin{cases} 1 \text{ if } \{V_i, V_j\} \text{ is an edge of the graph} \\ 0 \text{ if there is no edge joining } V_i \text{ and } V_j \end{cases}$$

ADJACENCY MATRIX of the graph: the matrix A where $A = (a_{ij})$
The matrix A called the adjacency matrix of the graph, is as follows:

$$A = \begin{pmatrix} 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&1 \\ 0&0&1&0&1 \\ 0&1&1&1&0 \end{pmatrix}$$ when $\{V_i, V_j\}$ edge on graph then $a_{ij} = a_{ji} = 1$, otherwise $a_{ij} = a_{ji} = 0$

From this we can conclude the following:
If A is an $n \times n$ matrix, of a graph and $a_{ij}^{(k)}$ represents the (i, j) entry of A^k, then $a_{ji}^{(k)}$ is equal to the number of walks of length k from V_i to V_j

Lecture 3

Equivalent systems:

Reason:
$A\mathbf{x} = \mathbf{b}$ which is an $m \times n$ linear system.
$\Leftrightarrow MA\mathbf{x} = M\mathbf{b} \Leftrightarrow M^{-1}(MA\hat{\mathbf{x}}) = M^{-1}(M\mathbf{b}) \Leftrightarrow A\hat{\mathbf{x}} = \mathbf{b} \Leftrightarrow U\mathbf{x} = \mathbf{c}$
Use of a sequence nonsingular matrices $E_1 \ldots E)k$ give us: $U = E_k \ldots E_1 A$ and $\mathbf{c} = E_k \ldots E_1 \mathbf{b}$
The system will be equivalent if M is nonsignular where $M = E_k \ldots E_1$
We know that M is nonsingular, because it is a product of nonsingular matrices.

Elementary matrix: a matrix that represents an elementary row operator.

Elementary matrices types:

Type	Definition	Example
1	obtained by interchanging 2 rows of I	$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ the first and second row of I_3 are swapped.
2	Obtained by multiplying a row by a nonzero number.	$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ last row of I_3 multiplied by 2
3	adding a multiple of one row to another, from matrix I	$E_3 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in I_3 : $(1) = (1) + 3(3)$

Theorem and Proof:

Elementary matrices nonsingularity E elementary matrix $\to E$ nonsingular $\to E^{-1}$ elementary matrix same type.

I proof this by example:

Type	calculations	conclusion
1	$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$= I_3 \to E_1^{-1} = E_1'$
2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$	$= I_3 \to E_2^{-1} = E_2'$
3	$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$= I_3 \to E_3^{-1} = E_3'$

Row equivalent

A matrix B is row equivalent to a matrix A if there exist elementary matrices $E_1, E_2, \ldots E_k$ s.t.
$B = E_k \cdot E_{k-1} \cdot E_2 \cdot E_1 A$

Theorem:

(1) When A is row equivalent to B then B is row equivalent to A
(2) Let A be a square matrix, then the following statements are equivalent:
(a) A is nonsingular
(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. $\mathbf{0}$
(c) A is row equivalent to I

Proof:

(1):
When A is row equivalent to B then:
$A = E_k \cdot E_{k-1} \cdot \ldots \cdot E_2 \cdot E_1 \cdot B$
$\to A \cdot E_k^{-1} = E_k \cdot E_k^{-1} \cdot E_{k-1} \cdot \ldots \cdot E_2 \cdot E_1 \cdot B$
$\to A \cdot E_k^{-1} = E_{k-1} \cdot \ldots \cdot E_2 \cdot E_1 \cdot B$
$\to A \cdot E_k^{-1} \cdot E_{k-1}^{-1} = E_{k-2} \cdot \ldots E_2 \cdot E_1 \cdot B$
Do this till the righthandside is equal to B
$A \cdot E_k^{-1} \cdot E_{k-1}^{-1} \cdot \ldots \cdot E_2^{-1} \cdot E_1^{-1} = B$
If A is rowequivalent to B and B is row equivalent to C then A is also row equivalent to C

(2):
If (a) holds, then (b) must hold.
Let x be s.t. $A\mathbf{x} = \mathbf{0}$ then, $\mathbf{x} = I\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$

If (b) holds, then (c) must holds.
(-)Applying elementary row operations to $A\mathbf{x} = \mathbf{0}$ we obtain $U\mathbf{x} = \mathbf{0}$ where U is in reduced row echelon form.
(-) If U has a zero row, there would be a nonzero solution, because there must be at least one free variable.
Therefore, U has no such rows, since it is in reduced row echelon form, we see that $U = I$

If (c) holds, then (a) must holds.
(-) $A = E_k E_{k-1} \ldots E_2 E_1 I = E_k E_{k-1} \ldots E_2 E_1$
(-) Elementary matrices are nonsingular and product of nonsingular matrices is nonsingular so A is also nonsingular.

Inverse computation application:

$A\mathbf{x} = \mathbf{b}$ then $\mathbf{x} = A^{-1}\mathbf{b}$

Observation:

Square matrix A nonsinuglar $\leftrightarrow$ I is row equivalent to A
$I = E_k E_{k-1} \ldots E_2 E_1 A \to E_1^{-1} E_2^{-1} \ldots E_{k-1}^{-1} E_k = A$
substitute this we find, $A(E_k E_{k-1} \ldots E_2 E_1) = I$

Conclusion and computation:

Conclusion: $A^{-1} = E_k E_{k-1} \ldots E_2 E_1$
Conclusion:
$AB = I \to BA = I$
Transform the matrix $\begin{bmatrix} A & I \end{bmatrix}$ into reduced row echelon form. If A is nonsingular, then you will obtain $\begin{bmatrix} I & A^{-1} \end{bmatrix}$ since:
$E_k E_{k-1} \ldots E_2 E_1 \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$

Example:

$A = \left(\begin{smallmatrix} 2 & 4 \\ 3 & 2 \end{smallmatrix}\right)$
$\left(\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right) \Rightarrow (1) = \frac{1}{2}(1) \Rightarrow \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 3 & 2 & 0 & 1 \end{array}\right) \Rightarrow (2) = -3(1) + 2$
$\Rightarrow \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -4 & -\frac{3}{2} & -3 \end{array}\right) \Rightarrow (2) = -\frac{1}{4}(2) \Rightarrow \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{8} & -\frac{3}{4} \end{array}\right) \Rightarrow (1) = -2(2){+}(1) \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & \frac{3}{2} \\ 0 & 1 & \frac{3}{8} & -\frac{3}{4} \end{array}\right)$ So A is nonsingular.

Application of nonsingularity to linear (square) systems:

The linear system $A\mathbf{x} = \mathbf{b}$ where $A = n \times n$ it has a unique solution $\leftrightarrow$ A is nonsingular.

Proof:

Proof 1:
Suppose A is nonsingular: $\mathbf{x}$ satisfies: $A\mathbf{x} = \mathbf{b}$ then $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \rightarrow \mathbf{x} = A^{-1}\mathbf{b}$ so unique solution.
Proof 2:
(-) Suppose that $\hat{\mathbf{x}}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$
(-) Let $\mathbf{z}$ be such that:$A\mathbf{z} = \mathbf{0}$ A is namely nonsingular if and only if $\mathbf{z} = \mathbf{0}$
(-) Note that $A(\hat{\mathbf{x}} + \mathbf{z}) = A\hat{\mathbf{x}} + A\hat{\mathbf{z}} = A\hat{\mathbf{x}} = \mathbf{b}$
(-) Uniqueness implies that $\hat{\mathbf{x}} + \mathbf{z} = \hat{\mathbf{x}}$ hence $\mathbf{z} = \mathbf{0}$
(-) Consequently, A is nonsingular.

Elementary triangular factorization:

A square matrix is said to be:
(-) upper triangular if $a_{ij} = 0$ for $i > j$
(-) lower triangular if $a_{ij} = 0$ for $i < j$
(-) Triangular if it is either upper triangular or lower triangular.
(-) Diagonal: If $a_{ij} = 0$ for $i \neq j$.
(-) strict upper (*lower*) triangular if it is upper (*lower*) triangular and every diagonal entry is nonzero.

Triangular (LU) factorization of elementary matrices:

If square matrix A can be geduced to strict upper triangular form by (3), then it can be written by an lower (L) and upper (U) triangular matrix $\Rightarrow$ factorization: *LU* factorization.

Example:

Strict upper triangular:
$\left(\begin{smallmatrix} 1 & 2 & 4 \\ 2 & 5 & 10 \\ 3 & 7 & 13 \end{smallmatrix}\right)$

$\left(\begin{smallmatrix} 1 & 2 & 4 \\ 2 & 5 & 10 \\ 3 & 7 & 13 \end{smallmatrix}\right)$	$\Rightarrow$ (2)=-2(1)+(2) (3)=-3(1)+(3) $\Rightarrow$	$\left(\begin{smallmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{smallmatrix}\right)$	$\Rightarrow (3) = -(2) + (3) \Rightarrow$	$\left(\begin{smallmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{smallmatrix}\right)$
$\left(\begin{smallmatrix} 1 & 2 & 4 \\ 2 & 5 & 10 \\ 3 & 7 & 13 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{smallmatrix}\right)$		$\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{smallmatrix}\right)$	$= \left(\begin{smallmatrix} 1 & 2 & 4 \\ 2 & 5 & 10 \\ 3 & 7 & 13 \end{smallmatrix}\right)$

So therefore we know:
$\left(\begin{smallmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) A$

Take the inverse of the first 3 matrices on the left hand side, and multiply everything out, then you get:
$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$
The left one is a lower triangular, and the right one is upper triangular.

Partioned matrices:

Rules:
(-) If $A_{m\times n}$ and $B_{n\times r}$ has been partitioned into columns $(\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_n)$ then $AB = (A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_r)$
(-) If $B = (B_1 \ B_2)$ where $(B_1)_{n\times t}$ and $(B_2)_{n\times(r-t)}$,and $A_{m\times n}$ then $AB = (AB_1 \ AB_2)$
(-) $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ where $(A_1)_{k\times n}$ and $(A_2)_{(m-k)\times n}$ then $AB = \begin{pmatrix} A_1B \\ A_2B \end{pmatrix}$
(-) $A = (A_1 \ A_2)$ and $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ then $AB = A_1B_1 + A_2B_2$
(-) $A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$ and $B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$ then: $AB = \left(\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right)$

Scaler product or inner product: $\mathbf{x}^T\mathbf{y}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
Inner product: $\mathbf{x}\mathbf{y}^T$ where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
Outer product expansion: XY^T where X is an $m \times n$ matrix and Y an $k \times n$ matrix.

Lecture 4

Determinants:

DEFINITION 1: $A = (a_{ij})$ which is $n \times n$. Then M_{ij} denote the $(n-1) \times (n-1)$ matrix, obtained by deleting the row and column of A which ocntains a_{ij}
DEFINITION 2: Determinant of $n \times n$ matrix $(\det(A))$ scaler defined by:

$$\det(A) = \begin{cases} a_{11} \text{ if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} \text{ if } n > 1 \end{cases}$$

Where $A_{ij} = (-1)^{i+j} \det(M_{ij})$

(-) $\det(M_{ij})$ minor of a_{ij}
(-) A_{ij} cofactor of a_{ij}

Example:

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ then $M_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$ and $M_{32} = \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}$

Cofactor expansion:

$A \in \mathbb{R}^{n \times n}$ with $n \geq 2$? $\det(A)$ by cofactor expansion along any row and column.
(-) Row: $\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}$
(-) Column: $\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}$

Examples:

$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$P = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$
Cofactor expansion along row 1	Cofactor expansion along row 1
$\det(F) = a_{11}A_{11} + a_{12}A_{12}$ $= (-1)^{1+1}a_{11}\det(M_{11}) + (-1)^{1+2}a_{12}\det(M_{12})$ $= (-1)^{1+1}a\det(d) + (-1)^{1+2}b\det(c)$	$\det(P) = (-1)^{1+1}a \begin{vmatrix} e & f \\ h & i \end{vmatrix} + (-1)^{1+2} \begin{vmatrix} d & f \\ g & i \end{vmatrix}$ $+(-1)^{1+3}c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$
$= ad - bc$	$= aei + bfg + cdh - gec - hfa - idb$
	Saurus' rule (only for $n = 3$)!!!!

Theorem:

Let A be a square matrix, Then, the following statements holds.
(1) $\det(A) = 0$ when:
(a) A has a zero row
(b) or A has a zero column
(c) or A has two identical columns
(d) or A has two identical rows.
(2) A triangular matrix?$\det(A) =$ product diagonal entries.

Example:

$T = \begin{pmatrix} 1&7&1&3\\ 3&0&0&1\\ 0&1&0&0\\ 1&9&3&2 \end{pmatrix}$

Cofactor expansion along row 3

$\det(T) = (-1)^{3+2}\det\begin{pmatrix} 1&1&3\\ 3&0&1\\ 1&3&2 \end{pmatrix} = -(-3\det\begin{pmatrix} 1&3\\ 3&2 \end{pmatrix} - 1\det\begin{pmatrix} 1&1\\ 1&3 \end{pmatrix}) = 3(2-9) + (3-1) = -19$

Lemma

Let A be a $n \times n$ matrix. If $i \neq j$ then $a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn} = 0$

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \ldots & a_{in} \\ \vdots & & \vdots \\ a_{j1} & \ldots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \to A^* = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \ldots & a_{in} \\ \vdots & & \vdots \\ a_{i1} & \ldots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$$

$0 = \det(A^*)$ 'cause it has two identity rows.

$0 = a_{i1}A^*_{j1} + a_{i2}A^*_{j2} + \ldots + a_{in}A^*_{jn} = a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn}$

Effect of elementary row operations:

Type	Rule	Calculations	conclusion
1	Two rows interchanged	$A = \begin{pmatrix} a&b\\ c&d \end{pmatrix}$,$E = \begin{pmatrix} 0&1\\ 1&0 \end{pmatrix}$ So $EA = \begin{pmatrix} c&d\\ a&b \end{pmatrix}$ $\det(EA) = ad - bc$ and $\det(EA) = bc - ad$	So $\det(EA) = -\det(A)$ $= \det(E)\det(A)$
2	One row is multiplied by a nonzero number	$\det(EA) = \alpha\det(A)$ $\det(E) = \det(EI) = \alpha\det(I) = \alpha$	$\det(EA) = \det(E)\det(A)$
3	Adding a multiple of one row to another row.	$\det(E) = I$ Expanding along jth row: $\det(EA) = (a_{j1} + ca_{i1})A_{i1} +$ $\ldots + (a_{jn} + ca_{in})A_{jn}$ $\det(EA) = \det(A) + c \cdot 0$	$\det(EA) = \det(A)$ $\det(EA) = \det(E)\det(A)$

$$\det(EA) = \det(E)\det(A) \begin{cases} \det(E) = -1 \\ \det(E) = \alpha \neq 0 \\ \det(E) = 1 \end{cases}$$

Examples:

Vandermonde matrices:

$v_3 = \begin{pmatrix} 1&a&a^2\\ 1&b&b^2\\ 1&c&c^2 \end{pmatrix}$ where $a \neq b \neq c$

$\det\begin{pmatrix} 1&a&a^2\\ 1&b&b^2\\ 1&c&c^2 \end{pmatrix} = \det\begin{pmatrix} b-a & b^2-a^2\\ c-ac & c^2-a^2 \end{pmatrix}$

$= (b-a)(c-a)\det\begin{pmatrix} 1 & b+a\\ 1 & c+a \end{pmatrix} = (b-a)(c-a)(c-b) = (a-b)(b-c)(c-a)$

Lecture 5

Singularity, row operations:

A square matrix A is singular iff $\det(A) = 0$

Proof:

Proof by contradiction:
B nonsingular $\Leftrightarrow U = E_k E_{l-1} \dots E_1 B$
Where U is in Reduced row echelon form and E_i all elementary matrices.
$\Leftrightarrow \det(U) = \det(E_k E_{k-1} \dots E_1 B) \Leftrightarrow \det(U) = \det(E_k)\det(E_{k-1})\dots\det(E_1)\det(B)$
We know that $\det(E_i) \neq 0$ so then $\det(B) \neq 0$ iff $\det(U) \neq 0$
$\Leftrightarrow U$ in reduced row echelon form, so $\det(U) \neq 0$ iff $U = I$
$\Leftrightarrow \det(A) \neq 0$ iff B row equivalent to $I \Leftrightarrow \det(B) \neq 0$ iff B is nonsingular.
Proven by contradiction.

observation:

Every square matrix can be transformed to row echelon form, that is:
$R = E_k E_{k-1} \dots E_1 A$
Where R is in row echelon form, and E_i's are all elementary matrices.
If the last row of R is zero, then $\det(A) = 0$
Otherwise, A is nonsingular and:
$\det(A) = [\det(E_k)\det(E_{k-1})\dots\det(E_1)]^{-1}$

Row operations vs cofactor expansion:

	Row operation		Cofactor expansion	
n	addition	multiplication	addition	mulitiplication
2	1	3	1	2
3	5	10	5	9
4	14	23	23	40
5	30	44	119	205
10	285	339	3628799	6235300

determinant of a product:

$A\&B$ both $n \times n$ matrices? Then $\det(AB) = \det(A)\det(B)$

Proof:

Case 1: B is singular:
$Bx = 0$ has nontrivial solution, so $ABx = 0$ has a nontrivial solution, so AB is singular,so:
$0 = \det(AB) = \det(A)\det(B) = 0$
Case 2: B is nonsingular:
B row equivalent to I so B is product of elementary matrices, so AB is a A times a product of elementary matrices. We know that $\det(ME) = \det(M)\det(E)$
$\det(AB) = \det(A)\det(E_k E_{k-1} \dots E_1) = \det(A)\det(B)$

Adjoint of a matrix

Let A be a $n\times n$ matrix. Its adjoint is defined by (where $A_{ij}=(-1)^{i+j}\det(M_{ij})$)

$\text{adj}A=\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn}\end{pmatrix}$

fact:

Let A be a $n\times n$ matrix then:

$a_{i1}A_{j1}+a_{i2}A_{j2}+\ldots+a_{in}A_{jn}=\begin{cases}\det(A)\,\text{if}\,i=j\\ 0\,\text{if}\,i\neq j\end{cases}$

observation:

$(A(\text{adj}A))_{ij}=a_{i1}A_{j1}+a_{i2}A_{j2}+\ldots+a_{in}A_{jn}$
$A(\text{adj}A)=\det(A)I$
$A=(\frac{1}{\det(A)}\text{adj}A)=I\,\text{if}\det(A)\neq 0$
$A^{-1}=\frac{1}{\det(A)}\text{adj}(A)\;\text{if}\det(A)\neq 0$

Example:

$A=\begin{pmatrix}2&1&0\\3&2&1\\1&2&3\end{pmatrix}$

$\text{adj}A=\begin{pmatrix}\begin{vmatrix}2&1\\2&3\end{vmatrix} & -\begin{vmatrix}1&0\\2&3\end{vmatrix} & \begin{vmatrix}1&0\\2&1\end{vmatrix}\\ -\begin{vmatrix}3&1\\1&3\end{vmatrix} & \begin{vmatrix}2&0\\1&3\end{vmatrix} & -\begin{vmatrix}-2&0\\3&1\end{vmatrix}\\ \begin{vmatrix}3&2\\1&2\end{vmatrix} & -\begin{vmatrix}2&1\\1&2\end{vmatrix} & \begin{vmatrix}2&1\\3&2\end{vmatrix}\end{pmatrix}=\begin{pmatrix}4&-3&1\\-8&6&-2\\4&-3&1\end{pmatrix}$

Cramer's rule:

Let $A\in\mathbb{R}^{n\times n}$ and $b\in\mathbb{R}^n$. Let A_i be the matrix obtained from A by replacing the ith column by $\mathbf{b}$. If $\mathbf{x}$ is the unique solution of $A\mathbf{x}=\mathbf{b}$, then $x_i=\frac{\det(A_i)}{\det(A)}$ for $i=1,2,\ldots,n$

Proof:

$A\mathbf{x}=\mathbf{b}\rightarrow\mathbf{x}=A^{-1}\mathbf{b}=\frac{1}{\det(A)}(\text{adj}A)b$
$x_i=\frac{b_1A_{1i}+b_2A_{2i}+\ldots+b_nA_{ni}}{\det(A)}=\frac{\det(A_i)}{\det(A)}$

Example:

$A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$ and $b=\begin{pmatrix}b_1\\b_2\end{pmatrix}$ where $A\mathbf{x}=\mathbf{b}$
If $a_{11}a_{22}-a_{12}a_{21}\neq 0$ then:
$x_1=\frac{\det\{A_1\}}{\det(A)}$ and $x_2=\frac{\det(A_2)}{\det(A)}$
$A_1=\begin{pmatrix}b_1&a_{12}\\b_2&a_{22}\end{pmatrix}$, $A_2=\begin{pmatrix}a_{11}&b_1\\a_{21}&b_2\end{pmatrix}$
$x_1=\frac{b_1a_{22}-b_2a_{12}}{a_{11}a_{a22}=a_{12}a_{21}}$ and $x_2=\frac{b_2a_{11}-b_1a_{21}}{a_{11}a_{a22}=a_{12}a_{21}}$
As we observed for Cramer's rule, requires computation of $n+1$ $n\times n$ determinants.

Lecture 6

Vector spaces:

V set, $\mathbb{F}$ set of scalers($\mathbb{R}$ or $\mathbb{C}$)
$\bigoplus : V \times V \to V$ and $\bigodot : \mathbb{F} \times V \to V$ respectively addition and scaler multiplication operators:
$\mathbf{x}, \mathbf{y} \in V \Rightarrow \mathbf{x} \bigoplus \mathbf{y} \in V$ and $\alpha \in \mathbb{F}, \mathbf{x} \in V \Rightarrow \alpha \bigodot \mathbf{x} \in V$

We say that $(V, \mathbb{F}, \bigodot, \bigoplus)$ form a vector space if the following axioms are satisfied.

Axiom		Proof
1	$\mathbf{x} \bigoplus \mathbf{y} = \mathbf{y} \bigoplus \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$	$\mathbf{x} = 1 \bigodot \mathbf{x}$ $\mathbf{x} = (1+0) \bigodot \mathbf{x}$ $\mathbf{x} = (1 \bigodot \mathbf{x}) \bigoplus (0 \bigodot \mathbf{x})$ $\mathbf{x} = \mathbf{x} \bigoplus (o \bigodot \mathbf{x})$ $-\mathbf{x} \bigoplus \mathbf{x} = -\mathbf{x} \bigoplus (\mathbf{x} \bigoplus (0 \bigodot \mathbf{x}))$ $-\mathbf{x} \bigoplus \mathbf{x} = (-\mathbf{x} \bigoplus \mathbf{x}) \bigoplus (0 \bigodot \mathbf{x})$ $\mathbf{x} \bigoplus -\mathbf{x} = (\mathbf{x} \bigoplus -\mathbf{x}) \bigoplus (0 \bigodot \mathbf{x})$ $\mathbf{0} = \mathbf{0} \bigoplus (0 \bigodot \mathbf{x})$ $\mathbf{0} = (0 \bigodot \mathbf{x})$
2	$(\mathbf{x} \bigoplus \mathbf{y}) \bigoplus \mathbf{z} = \mathbf{x} \bigoplus (\mathbf{y} \bigoplus \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$	$-\mathbf{x} = -\mathbf{x} \bigoplus \mathbf{0}$ $-\mathbf{x} = -\mathbf{x} \bigoplus (\mathbf{x} \bigoplus \mathbf{y})$ $-\mathbf{x} = (-\mathbf{x} \bigoplus \mathbf{x}) \bigoplus \mathbf{y}$ $-\mathbf{x} = (\mathbf{x} \bigoplus -\mathbf{x}) \bigoplus \mathbf{y}$ $-\mathbf{x} = \mathbf{0} \bigoplus \mathbf{y}$ $-\mathbf{x} = \mathbf{y} \bigoplus \mathbf{0}$ $-\mathbf{x} = \mathbf{y}$
3	There exists $\mathbf{0} \in V$ s.t. $\mathbf{x} \bigoplus \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$	
4	For each $\mathbf{x} \in V$ there exists $-\mathbf{x} \in V$ s.t. $\mathbf{x} \bigoplus -\mathbf{x} = \mathbf{0}$	
5	$\alpha \bigodot (\mathbf{x} \bigoplus \mathbf{y}) = (\alpha \bigodot \mathbf{x}) \bigoplus (\alpha \bigodot \mathbf{y})$ for all $\alpha \in \mathbb{F}$ and $\mathbf{x}, \mathbf{y} \in V$	
6	$(\alpha + \beta) \bigodot \mathbf{x} = (\alpha \bigodot \mathbf{x}) \bigoplus (\beta \bigodot \mathbf{x})$ for all $\alpha, \beta \in \mathbb{F}$ and $\mathbf{x} \in V$	
7	$(\alpha \cdot \beta) \bigodot \mathbf{x} = \alpha \bigodot (\beta \bigodot \mathbf{x})$ for all $\alpha, \beta \in \mathbb{F}$ and $\mathbf{x} \in V$	
8	$1 \bigodot \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$	$\mathbf{0} = 0 \bigodot \mathbf{x}$ $\mathbf{0} = (1 + (-1)) \bigodot \mathbf{x}$ $\mathbf{0} = (1 \bigodot \mathbf{x}) \bigoplus ((-1) \bigodot \mathbf{x})$ $\mathbf{0} = \mathbf{x} \bigoplus ((-1) \bigodot \mathbf{x})$ $(-1) \bigodot \mathbf{x} = -\mathbf{x}$

$\mathbf{x} \in V$ then: $0 \bigodot \mathbf{x} = \mathbf{0}$ $\quad \mathbf{x} \bigoplus -\mathbf{x} = \mathbf{0} \Rightarrow -\mathbf{x} = -\mathbf{x}$ $\quad (-1) \bigodot \mathbf{x} = -\mathbf{x}$

Examples:

1:
Let $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ Define $x \bigoplus y := x + y$ and $\alpha \bigodot x := \alpha x$
2:
Let $A, B \in \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}$. Define $A \bigoplus B := A + B$ and $\alpha \bigodot A := \alpha A$
3:
Let $a, b \in \mathbb{R}$ with $a < b$. Define $C[a, b] := \{f : [a, b] \to \mathbb{R} | f \text{is continuous}\}$
Let $f, g \in C[a, b]$ and $\alpha \in \mathbb{R}$
$(f \bigoplus g)(x) := f(x) + g(x)$ for all $x \in [a, b]$
$(\alpha \bigodot f)(x) := \alpha f(x)$ for all $x \in [a, b]$
4:
Let n be a positive integer. Define $P_n = \{p | p \text{ is a polynomial of degree less than } n\}$
Let $p, q \in P_n$ and $\alpha \in \mathbb{R}$
$(p \bigoplus q)(x) := p(x) + q(x)$
$(\alpha \bigodot p)(x) := \alpha p(x)$

Special types of vector spaces:

$C[a, b]$: Set all real-valued functions that are defined and continuous on $[a, b]$.
P_n is the set of all polynomials of degree less than n

The nullspace of a matrix

$N(A) = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{0}\}$
If the linear system $A\mathbf{x} = \mathbf{b}$ is consistent and $\mathbf{x}_0$ particular solution, then the vector $\mathbf{y}$ will also be a solution iff $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$ where $\mathbf{z} \in N(A)$

Supspace

From now one, we write $\mathbf{x} + \mathbf{y}$ and $\alpha\mathbf{x}$ meaning $\mathbf{x} \bigoplus \mathbf{y}$ and $\alpha \bigodot \mathbf{x}$ respectively.

Let S be a subset of vector space V. We say that S is a subspace of V if:
(-) S is nonempty.
(-) $\mathbf{x} \in S$ and ,$\alpha \in \mathbb{F} \Rightarrow \alpha\mathbf{x} \in S$
(-) $\mathbf{x}, \mathbf{y} \in S \Rightarrow \mathbf{x} + \mathbf{y} \in S$
Trivial subspaces $\{\mathbf{0}\}$ and V are subspaces of V
All other subspaces of V are refereed to as proper subspaces.
We refer to $\{\mathbf{0}\}$ as the zero subspace.
If S is a subspace, then $\mathbf{0} \in S$

Example:

1:
Let $S = \{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3 | x_1 = x_2\}$
$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S \Rightarrow S$ is nonempty.
Let $x \in S$. Then there are real numbers a and b s.t. $\mathbf{x} = \begin{pmatrix} a \\ a \\ b \end{pmatrix}$.
Let $\mathbf{x} = \begin{pmatrix} a \\ a \\ b \end{pmatrix} \in S$ and $\alpha \in \mathbb{R}$. Then, $\alpha\mathbf{x} = \alpha \begin{pmatrix} a \\ a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha a \\ \alpha b \end{pmatrix} \in S$
Let $\mathbf{x} = \begin{pmatrix} a \\ a \\ b \end{pmatrix} \in S$ and $\mathbf{y} = \begin{pmatrix} c \\ c \\ d \end{pmatrix} \in S$. Then $\mathbf{x} + \mathbf{y} = \begin{pmatrix} a \\ a \\ b \end{pmatrix} + \begin{pmatrix} c \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ a+c \\ b+d \end{pmatrix} \in S$

2:
$S = \{A \in \mathbb{R}^{2\times2} | a_{12} = -a_{21}\}$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S \Rightarrow S$ is nonempty.
Let $A \in S$ then there are real numbers a, b, c s.t. $A = \begin{pmatrix} a & b \\ -b & c \end{pmatrix}$
Let $A = \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \in S$ and $\alpha \in \mathbb{R}$ then $\alpha A = \alpha \begin{pmatrix} a & b \\ -b & c \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ -\alpha b & \alpha c \end{pmatrix} \in S$
Let $A = \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \in S$ and $B = \begin{pmatrix} d & e \\ -e & f \end{pmatrix} \in S$ then: $A + B = \begin{pmatrix} a & b \\ -b & c \end{pmatrix} + \begin{pmatrix} d & e \\ -e & f \end{pmatrix} = \begin{pmatrix} a+d & b+e \\ -b-e & c+f \end{pmatrix} \in S$

3: $S = \{p \in P_n | p(1) = 0\}$ and $n > 0$
$p(x) = x - 1 \in S \Rightarrow S$ is nonempty.
Let $p \in S$ and $\alpha \in \mathbb{R}$,then $(\alpha p)(1) = \alpha p(1) = 0 \Rightarrow \alpha p \in S$
Let $p, q \in S$ then
$(p + q)(1) = p(1) + q(1) = 0 \Rightarrow p + q \in S$
4:
Let $S \subseteq C[a, b]$ be the set of all functions that have a continuous derivative on $[a, b]$
$f(x) = 1$ for all $x \in [a, b] \Rightarrow f \in S \Rightarrow S$ is nonempty.
Let $f \in S$ and $\alpha \in \mathbb{R}$ then: $(\alpha f)' = \alpha f' = \alpha f \in S$
Let $f \in S$ and $g \in S$ then: $(f + g)' = f' + g' \Rightarrow f + g \in S$
5:
$S = \{f \in C[-1, 1] | f(-x) = -f(x)$ for all $x \in [-1, 1]\}$
$f(x) = 0$ for all $x \in [a, b] \Rightarrow f \in S \Rightarrow S$ is nonempty.
Let $f \in S$ and $\alpha \in \mathbb{R}$ then:
$(\alpha f)(-x) = \alpha f(-x) = -\alpha f(x) = -(\alpha f)(x)$ for all $x \in [-1, 1] \Rightarrow \alpha f \in S$.
Let $f \in S$ and $g \in S$ then:
$(f + g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f + g)(x)$ for all $x \in [-1, 1] \Rightarrow f + g \in S$

6: NOT
$S := \{\mathbf{x} \in \mathbb{R}^2 | x_2 = x_1^2\}$
$\mathbf{0} \in S \Rightarrow S$ is nonempty.
Let $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S$ and $2 \in \mathbb{R}$ but
$2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \notin S$
Let $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix} \in S$, but:
$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \notin S$

7:
Let $A \in \mathbb{R}^{m\times n}$ define: $N(A) := \{\mathbf{x} \in R^n | A\mathbf{x} = \mathbf{0}\}$
$\mathbf{0} \in N(A) \Rightarrow N(A)$ is nonempty.
Let $\mathbf{x} \in N(A)$ and $\alpha \in \mathbb{R}$ then: $A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \mathbf{0} \Rightarrow \alpha\mathbf{x} \in N(A)$
Let $\mathbf{x} \in N(A)$ and $\mathbf{y} \in N(A)$. then: $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} \rightarrow \mathbf{x}+\mathbf{y} \in N(A)$

8:
Let $A = \left(\begin{smallmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{smallmatrix}\right)$. To determine $N(A)$ we have to solve $A\mathbf{x} = \mathbf{0}$. Using Gauss-Jordan elimination, we get:
$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right) \Rightarrow (2) = (2) - 2(1) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right) \Rightarrow (2) = -(2) \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right) \Rightarrow$
$(1) = (1) - (2) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right)$
$x_1 = x_3 - x_4$ and $x_2 = -2x_3 + x_4$
So $N(A) = \{a \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} | a, b \in \mathbb{R}\}$

Lecture 7

Span:

LINEAR COMBINATION:
$\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ where V, vector space, $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ then $\alpha_1\mathbf{v}_1 + \ldots + \alpha_n\mathbf{v}_n$ the linear combination.
SPAN OF VECTORS:
Set of all linear combinations of the given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$
Denoted by $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ or $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \{\alpha\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n | \alpha_1, \ldots, \alpha_n \in \mathbb{F}\}$
In these cases:
(-) $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$
Or $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spanning set for V we say $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span V

Example:

$N(A) = \{a \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} | a, b \in \mathbb{R}\} = \text{span}(\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix})$

Theorem:

Let V be a vector space $\mathbf{v}_1, \ldots \mathbf{v}_n \in V$, the $\text{span}(\mathbf{v}_1, \ldots \mathbf{v}_n)$ is a subspace.

Proof:

(-) By choosing $\alpha_1 = \ldots = \alpha_n = 0$ we get $\mathbf{0} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$
$\Rightarrow \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) \neq \emptyset$

(-) Let β a scaler and $\mathbf{v} \in \text{span}(\mathbf{v}_1, \ldots \mathbf{v}_n)$
(-) $\mathbf{v} = \alpha_1\mathbf{v}_1 + \ldots + \alpha_n\mathbf{v}_n$ for some scalers $\alpha_1, \alpha_2, \ldots, \alpha_n$
(-) $\beta\mathbf{v} = \beta\alpha_1\mathbf{v}_1 + \ldots + \beta\alpha_n\mathbf{v}_n \in \text{span}(\mathbf{v}_1, \ldots \mathbf{v}_n)$

(-) Let $\mathbf{v} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $w \in \text{span}(\mathbf{w}_1, \ldots, \mathbf{v}_n)$
(-) $\mathbf{v} = \alpha_1\mathbf{v}_1 + \ldots + \alpha_n\mathbf{v}_n \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$
(-) $\mathbf{w} = \beta_1\mathbf{v}_1 + \ldots + \beta_n\mathbf{v}_n \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$
(-) $\mathbf{v} + \mathbf{w} = (\alpha_1 + \beta_1)\mathbf{v}_1 + \ldots + (\alpha_n + \beta_n)\mathbf{v}_n \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$

Definition:

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for V if every vector of V can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

example:

1:
The set $\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\}$ is a spanning set for $\mathbb{R}^3$

$\begin{pmatrix}a\\b\\c\end{pmatrix}=\alpha_1\begin{pmatrix}1\\0\\0\end{pmatrix}+\alpha_2\begin{pmatrix}1\\1\\0\end{pmatrix}+\alpha_3\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\alpha_1+\alpha_2+\alpha_3=a,\alpha_2+\alpha_3=b,\alpha_3=c$
$\alpha_3=c,\alpha_2=b-c,\alpha_1=a-b$

2:
The set $\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\}$ is NOT a spanning set for $\mathbb{R}^3$:
$\begin{pmatrix}a\\b\\c\end{pmatrix}=\alpha_1\begin{pmatrix}1\\0\\1\end{pmatrix}+\alpha_2\begin{pmatrix}0\\1\\0\end{pmatrix}\Rightarrow\alpha_1=a,\alpha_2=b,\alpha_1=c$

3:
The set $\{1-x^2,x+2,x^2\}$ is a spanning set P_3
P_3 the set of polynomials with $n<3$
$ax^2+bx+c\in P_3$
$ax^2+bx+c=\alpha_1(1-x^2)+\alpha_2(x+2)+\alpha_3x^2$
$=(\alpha_3-\alpha_1)x^2+\alpha_2x+(\alpha_1+2\alpha_2)$
So: $a=\alpha_3-\alpha_1$,$b=\alpha_2$ and $c=\alpha_1+\alpha_2$
$\Rightarrow$ $\alpha_1,\alpha_2\&\alpha_3$ in terms of $a,b,c\Rightarrow$ spanning set.

Theorem:

Let $\mathbf{v}_1,\ldots,\mathbf{v}_n$ belong to a vector space V
(1) If $\mathbf{v}_1,\ldots,\mathbf{v}_n$ span V and one of them can be written as a linear combination of the other $n-1$ vectors, then those $n-1$ vectors span V
(2) one of the vectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$ is a linear combination of the other $n-1$ vectors iff there exist scalers $c_1,c_2,\ldots,c_n$ not all zero s.t. $c_1\mathbf{v}_1+\ldots+c_n\mathbf{v}_n=\mathbf{0}$

Proof:

(1) Suppose $\mathbf{v}_n$ linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_{n-1}$ and $\mathbf{x}\in V$ then:
$\mathbf{v}_n=\beta_1\mathbf{v}_1+\ldots+\beta_{n-1}\mathbf{v}_{n-1}$
$\mathbf{x}=\alpha_1\mathbf{v}_1+\ldots+\alpha_n\mathbf{v}_n$
$\mathbf{x}=\alpha_1\mathbf{v}_1+\ldots+\alpha_n(\beta_1\mathbf{v}_1+\ldots+\beta_{n-1}\mathbf{v}_{n-1})$
$\mathbf{x}=(\alpha_1+\alpha_n\beta_1)\mathbf{v}_1+\ldots+(\alpha_{n-1}+\alpha_n\beta_{n-1})\mathbf{v}_{n-1}$

(2)(a) Suppose $\mathbf{v}_n$ linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_n$ then:
$\mathbf{v}_n=\beta_1\mathbf{v}_1+\ldots+\beta_{n-1}\mathbf{v}_{n-1}\Rightarrow\beta_1\mathbf{v}_1+\ldots+\beta_{n-1}\mathbf{v}_{n-1}-\mathbf{v}_n=\mathbf{0}$
(b) $c_1\mathbf{v}_1+\ldots+c_n\mathbf{v}_n=\mathbf{0}$ and $c_n\neq0$
$\mathbf{v}_n=-\frac{1}{c_n}(c_1\mathbf{v}_1+\ldots+c_{n-1}\mathbf{v}_{n-1})$

Linear independence:

The vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n$ in a vector space V are said to be
LINEARLY INDEPENDENT if $c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n=\mathbf{0}$ implies $c_1=c_2=\ldots=c_n=0$
LINEARLY DEPENDENT if there exists scalers $c_1,c_2,\ldots c_n$, not all zero s.t. $c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n=\mathbf{0}$
Let $\mathbf{x}_1,\ldots,\mathbf{x}_n$ be n vectors in $\mathbb{R}^n$ let $X=(\,\mathbf{x}_1\,\cdots\,\mathbf{x}_n\,)$ then these vectors $\mathbf{x_1},\ldots,\mathbf{x}_n$ linearly dependent iff X is singular.

Theorem:

Let $f_1, f_2, \ldots, f_n$ functions in $C^{(n-1)}[a,b]$ and define $W[f_1, f_2, \ldots, f_n]$ on $[a,b]$ by:

$$W[f_1, f_2, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}$$

The function $W[f_1, f_2, \ldots, f_n](x)$ called Wronskian of $f_1, f_2, \ldots, f_n$

If there exists $x_0 \in [a,b]$ s.t. $W[f_1, f_2, \ldots, f_n](x_0) \neq 0$ then $f_1, \ldots, f_n$ linearly independent.

Theorem:

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be vector in $\mathbb{R}^n$ and let $X = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$. The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ linearly dependent iff X singular.

Proof:

$$c_1\mathbf{x}_1 + \ldots c_n\mathbf{x}_n = \mathbf{0} \Leftrightarrow \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0} \Leftrightarrow X \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}$$

Example:

1:
The vectors $\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}\} \in \mathbb{R}^3$ are linearly independent:
$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c_1 = 0, c_2 = 0$
2:
The vectors $p_1(x) = x^2 - 2x + 3$,$p_2(x) = 2x^2 + x + 8$,and $p_3(x) = x^2 + 8x + 7$ in P^3 are linearly dependent:
$c_1(x^2 - 2x + 3) + c_2(2x^2 + x + 8) + c_3(x^2 + 8x + 7) - 0 \cdot x^2 + 0 \cdot x + 0$
$(c_1 + 2c_2 + c_3)x^2 + (-2c_1 + c_2 + 8c_3)x + (3c_1 + 8c_2 + 7c_3) = 0 \cdot x^2 + 0 \cdot x + 0$
$\Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c_1 = 3, c_2 = -2, c_3 = 1$

Theorem:

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in vector space V. Every vector in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ can be written uniquely as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ iff $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

Proof:

Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ lin. independent, and let $x \in$span$(\mathbf{v}_1, \ldots, \mathbf{v}_n)$
So $\mathbf{x} = \alpha_1\mathbf{v} + \ldots + \alpha_n\mathbf{v}_n = \beta_1\mathbf{v}_1 + \ldots + \beta_n\mathbf{v}_n$
$\Rightarrow (\alpha_1 - \beta_1)\mathbf{v}_1 + \ldots + (\alpha_n - \beta_n)\mathbf{v}_n = \mathbf{0}$
$\mathbf{v}_1, \ldots, \mathbf{v}_n$ lin. independent $\Rightarrow \alpha_1 = \beta_i$ for all $1 \leq i \leq n$
$\Rightarrow$ unique combination.

Basis and dimension:

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a basis for a vector space V if:
(1) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.
(2) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span V

Example:

1:
The set $\{e_1, e_2, e_3\}$ where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,$e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, is a basis for $\mathbb{R}^3$. There are many other basis. For instance, the following are bases for $\mathbb{R}^3$
$\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$ and $\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 9 \end{pmatrix}\}$
2:
The set $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ where $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a basis for $\mathbb{R}^{2\times 2}$

Theorem:

Linearly dependent:
let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for a vector space V and m be a positive integer with $m > n$. then any collection of m vectors in V is linearly dependent.
Two basis:
If Both $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ bases for a vector space V then $n = m$

Proof

Linearly dependent.
Let $u_1, u_2, \ldots, u_m$ be m vectors in V where $m > n$ since $\{v_1, v_2, \ldots, v_n\}$ is a spanning sat we have:
$\mathbf{u}_1 = a_{11}\mathbf{v}_1 + \ldots + a_{1n}\mathbf{v}_n$
$u_1 = a_{21}v_1 + a_{22}v_2 + \ldots + a_{2n}v_n$
$\vdots$
$\mathbf{u}_m = a_{m1}\mathbf{v}_1 + \ldots + a_{mn}\mathbf{v}_n$
$0 = c_1\mathbf{u}_1 + \ldots + c_m\mathbf{u}_m = (\sum_{i=1}^{m} a_{il}c_i)\mathbf{v}_1 + (\sum_{i=1}^{m} a_{il}c_i)\mathbf{v}_n$
homogeneus $m \times n$ lin. system, with c_i unknowns: $\sum_{i=1}^{m} a_{ij}c_i = 0$,$j = 1, 2, \ldots, n$
Since $m > n$ there exists scalers $c_1, c_2, \ldots, c_n$ such that: $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_m\mathbf{u}_m = \mathbf{0}$
If both $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ are bases for a vector space V, then $n = m$
The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set and vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are linearly independent, As such Thm (4.45) implies that $m \leq n$ same reasoning gives $n \leq m$ so $m \leq n$

Definition:

V vector space.
(-) basis n vecotrs $\Rightarrow \dim(V) = n$
(-) Subspace $S = \{\mathbf{0}\}$ of V $\dim(S) = 0$
(-) V finite dimensional $\Rightarrow$ finite set of vectors spans V. Otherwise infinite dimensional.

Lecture 8

Basis and dimension:

Let V be a vector space of dimension $n > 0$ then:
(1) any set of n vectors
(a) spans V are lin. independent.
(b) that are lin. independent spans V
(2) when $\#vectors < n$ cannot span V
(3) $\#vectors < n$ can be completed to form basis for V

Definition:

V vector space and $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ ordered basis for V
$\mathbf{x} \in V$ and $c_1, \ldots, c_n$ scalers then: $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$
So each $\mathbf{x}$ unique vector: COORDINATE VECTOR: $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$
COORDINATES OF $\mathbf{x}$ RELATIVE TO E: $c_1, \ldots, c_n$

Example:

In a certain town, 30% of the married women get divorced each year and 20% of the single womeng et married each year. There are 8000 married women and 2000 single women. Assume that the total populaition remains constant. How many married women and signle women will be after n years
Let m_k denote the number of married women year k and s_k the number of single women in year k
$\begin{pmatrix} m_{k+1} \\ s_{k+1} \end{pmatrix} = \begin{pmatrix} 0.7m_k + 0.2s_k \\ 0.3m_k + 0.8s_k \end{pmatrix} = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} m_k \\ s_k \end{pmatrix} \begin{pmatrix} m_0 \\ s_0 \end{pmatrix}$

$\begin{pmatrix} m_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 6000 \\ 4000 \end{pmatrix}$
When we go further we will see that:
$\begin{pmatrix} m_n \\ s_n \end{pmatrix} = \begin{pmatrix} 0{,}7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}^n \begin{pmatrix} m_0 \\ s_0 \end{pmatrix}$

Note that:
$\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0.5 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
This means that:
$\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}^n \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}^n \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0.5^n \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

So note that:
$\begin{pmatrix} m_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = 8000 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2000 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2000 \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 4000 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
As such we obtain:
$\begin{pmatrix} m_n \\ s_n \end{pmatrix} = 2000 \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 4000 \cdot (0.5)^n \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Change of basis:

$$E = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) \qquad F = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$$
$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n \qquad \mathbf{x} = d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + \ldots + d_n\mathbf{w}_n$$
$$\mathbf{c} = [\mathbf{x}]_E \qquad \mathbf{d} = [\mathbf{x}]_F$$
$$\mathbf{v}_1 = t_{11}\mathbf{w}_1 + t_{21}\mathbf{w}_2 + \ldots + t_{n1}\mathbf{w}_n$$
$$\mathbf{v}_2 = t_{12}\mathbf{w}_1 + t_{22}\mathbf{w}_2 + \ldots + t_{n2}\mathbf{w}_n$$
$$\vdots$$
$$\mathbf{v}_n = t_{1n}\mathbf{w}_1 + t_{2n}\mathbf{w}_2 + \ldots + t_{nn}\mathbf{w}_n$$

So we can say:
$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$
$\mathbf{x} = [\sum_{j=1}^{n} t_{1j}c_j]\mathbf{w}_1 + [\sum_{j=1}^{n} t_{2j}c_j]\mathbf{w}_2 + \ldots + [\sum_{j=1}^{n} t_{nj}c_j]\mathbf{q}_n$
$\Rightarrow d_i = \sum_{j=1}^{n} t_{ij}c_j \Rightarrow \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \ldots & t_{1n} \\ t_{21} & t_{22} & \ldots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{n1} & t_{n2} & \ldots & t_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$
$\mathbf{d} = T\mathbf{c}$ and T is called the transition matrix.

Example:

1:
Let $E = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix})$ and $F = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = (\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix})$

$\mathbf{v}_1 = t_{11}\mathbf{w}_1 + t_{21}\mathbf{w}_2 + t_{31}\mathbf{w}_3 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\mathbf{v}_2 = t_{12}\mathbf{w}_1 + t_{22}\mathbf{w}_2 + t_{32}\mathbf{w}_3 \Rightarrow \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\mathbf{v}_3 = t_{13}\mathbf{w}_1 + t_{23}\mathbf{w}_2 + t_{33}\mathbf{w}_3 \Rightarrow \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = -3 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} 0 \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

So $T = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix}$

2:
Let $E = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (1, 2x, 4x^2 - 2)$ and $F = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = (1, x, x^2)$

$\mathbf{v}_1 = t_{11}w_1 + t_{21}w_2 + t_{31}w_3 \Rightarrow 1 = 1 \cdot 1 + 0cdotx + 0 \cdot x^2$

$\mathbf{v}_2 = t_{12}w_1 + t_{22}w_2 + t_{32}w_3 \Rightarrow 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$

$\mathbf{v}_3 = t_{13}w_1 + t_{23}w_2 + t_{33}w_3 \Rightarrow 4x^2 - 2 = -2 \cdot 1 + 0 \cdot x + 4 \cdot x^2$

So $T = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow \mathbf{d} = [p(x)]_F = T[p(x)]_E = T\mathbf{c}$
$p(x) = 1 + 4x + 8x^2 \Rightarrow \mathbf{d} = \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}$ and $\mathbf{c} = T^{-1}\mathbf{d} = \begin{pmatrix} 1 & 0 & 0.5 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.25 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}$
$p(x) = 1 + 4x + 8x^2 = 5 \cdot 1 + 2 \cdot 2x + 2 \cdot (4x^2 - 2)$

Row and column space:

Let $A \in \mathbb{R}^{m\times n}$. Row space of A the subspace of $\mathbb{R}^{1\times n}$ spanned by the rows of A
Column space of A the subspace of $\mathbb{R}^{m\times 1}$ spanned by the columns of A

Example:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$
The row space of A consists of row vectors of the form:
$\alpha \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta & 2\alpha \end{bmatrix} \Rightarrow$rowspace$(A) =$span$(\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \end{pmatrix})$

Whereas the column space of A consists of column vectors of the form:
$\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta + 2\gamma \end{bmatrix} \Rightarrow$colspace$(A) =$span$(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix})$

row equivalent matrices and proof:

Two row equivalent matrices have the same row space.
Proof:
$B = AE_k E_{k-1} \dots E_1$ therefore rowspace$(A) \subseteq$rowspace(B)
$A = BE_l E_{l-1} \dots E_1$ therefore rowspace$(B) \subseteq$rowspace(A)
So rowspace$(A) =$rowspace(B)

Rank:

Rank of matrix A(rank(A)) =dimension row space A
determine rank A? first A in row echelon form.
nonzero rows echelon form=basis row space, number nonzero rows=rank.

(1) A linear system $A\mathbf{x} = \mathbf{b}$ is consistent iff $\mathbf{b}$ is a linear combination of the columns of A
(2) A linear system $A\mathbf{x} = \mathbf{b}$ is consistent iff $\mathbf{b}$ is in the column space of A

Proof:

Suppose $A\mathbf{x} = \mathbf{b}$ is consistent, then $\exists x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ s.t. $A \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$
given $x_1a_1 + x_2a_2 + \ldots + x_na_n = b \Rightarrow \mathbf{b}$ lim. com. of columns of A

$\mathbf{b}$ is a lin. col. of col. of A says: $\mathbf{b} = x_1a_1 + x_2a_2 + \ldots + x_na_n$
Where $x_1, x_2, \ldots, x_n$ are scalers and $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$, then: $A\mathbf{x} = \mathbf{b}$

Theorem:

Let $A \in \mathbb{R}^{m\times n}$, then:
(1) The linear system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$ iff the column vectors of A span $\mathbb{R}^m$
(2) The linear system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b} \in \mathbb{R}^m$ iff column vectors of A independent.

Proof

2:
$A\mathbf{x} = \mho$ at most 1 solution for every $\mathbf{b} \in \mathbb{R}^m \Rightarrow a\mathbf{x} = \mathbf{0}$ at most one solution $\Rightarrow$ lin. independent.
$\Rightarrow A\mathbf{x} = \mathbf{0}$ trivial solution.
Suppose $\mathbf{x}_1 \& \mathbf{x}_2$ solutions $A\mathbf{x}_1 =$
$\Rightarrow A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{b} - \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{x}_1 = \mathbf{x}_2$

Definition and theorems:

Dimension of the null space of A is called the nullity of A ; null(A)
Thm: $A \in \mathbb{R}^{m\times n}$ then rank(A)+null(A) $= n$

Proof

If $A \in \mathbb{R}^{m\times n}$, then rank($A$)+null($A$) $= n$
(-) Let U be a row reduced echelon form of A
(-) $A\mathbf{x} = \mathbf{0}$ iff $U\mathbf{x} = \mathbf{0}$
(-) If rank(A) $= r$ then U has r nonzero rows.
(-) Therefore, there are r lead variables and $n - r$ free variables.
(-) The dimension of the nullspace of A must be $n - r$

Theorem:

(1) An $n \times n$ matrix A nonsingular iff column vectors A form basis $\mathbb{R}^n$
(2) For every matrix, the dimension of the row space and that of the column space are equal.

Proof:

(-) Let $A \in \mathbb{R}^{m\times n}$,rank($A$) $= r$, and U be a row echelon form of A
(-) U has r leading 1's. and its columns corresponding to leading 1's are lin. independent.
(-) The column spaces of A and U are not the same, in general.
(-) Let $\hat{U} \in \mathbb{R}^{m\times r}$ be the matrix obtained form U by deleting all columns corresponding to free variables
(-) Let $\hat{A} \in \mathbb{R}^{m\times r}$ be the matrix obtained from A by deleting the same columns.
(-) $\hat{A}\mathbf{x} = \mathbf{0}$ iff $\hat{U}\mathbf{x} = \mathbf{0}$
(-) Since columns $\hat{U}$ linearly independent: $\hat{U}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$ Therefore: $\hat{A}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$
(-) As such the columns are linearly independent.
(-)dim(colspace(A)) $\geq$dim(colspace(A)) $\geq r =$ dim(rowspace(A))
(-) dim(rowspace(A))=dim(colspace(A^T)) $\geq$dim(rowspace(A^T)) =dim(colspace(A))
(-) dim(colspace(A)) =dim(rowspace(A))

Lecture 9

Linear transformation:

A mapping L from a vector space V into a vector space W is said to be a linear transformation if:
$L(\alpha\mathbf{x}+\beta\mathbf{y})=\alpha L(\mathbf{x})+\beta L(\mathbf{y})$ for all vectors $x,y\in L\in V$ and scalers α,β
where the leftpart is of V the right part of W
TERMINOLOGY:
(1) $L:V\to W$ a mapping L from a vector space V into a vector space W
(2) $L:V\to V\Rightarrow L$ is an operator.

Example:

1:
The operator $L:\mathbb{R}^2\to\mathbb{R}^2$ given by $L(\mathbf{v})=\begin{bmatrix}-v_2\\-v_1\end{bmatrix}$ for $\mathbf{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$ is linear.
$L(\alpha\mathbf{x}+\beta\mathbf{y})=L(\alpha\begin{bmatrix}x_1\\x_2\end{bmatrix}+\beta\begin{bmatrix}y_1\\y_2\end{bmatrix})=L(\begin{bmatrix}\alpha x_1+\beta y_1\\\alpha x_2+\beta y_2\end{bmatrix})=\begin{bmatrix}-\alpha x_2-\beta y_2\\\alpha x_1+\beta y_1\end{bmatrix}$
$L(\alpha\mathbf{x}+\beta\mathbf{y})=\alpha\begin{bmatrix}-x_2\\x_1\end{bmatrix}+\beta\begin{bmatrix}-y_2\\y_1\end{bmatrix}=\alpha L(\curvearrowleft)+\beta L(\mathbf{y})$
2:
$L:C[a,b]\to\mathbb{R}$ given by $L(f)=\int\limits_a^b f(x)dx$ linear transformation:
$L(\alpha f+\beta g)=\int\limits_a^b(\alpha f+\beta g)(x)dx=\alpha\int\limits_a^b f(x)dx+\beta\int\limits_a^b g(x)dx=\alpha L(f)+\beta L(g)$
3:
$D:C^1[a,b]\to C[a,b]$ given by $D(f)=f'$ linear transformation.

$C^1[a,b]=\{f:[a,b]\to\mathbb{R}|f'\in C[a,b]\}$
$D(\alpha f+\beta g)=(\alpha f+\beta g)'=\alpha f'+\beta g'=\alpha D(f)+\beta D(g)$
4:
The mapping $L:P_4\to P_4$ given by $L(p)=p(3)x^3+p(2)x^2+p(1)x+p(0)$ is linear.

$L(\alpha p+\beta q)=(\alpha+\beta q)(3)\cdot x^3+(\alpha p+\beta q)(2)\cdot x^2+(\alpha p+\beta q)(1)\cdot x+(\alpha p+\beta q)(0)$
$=\alpha(p(3)x^3+p(2)x^2+p(1)x+p(0))+\beta(q(3)x^3+q(2)x^2+q(1)x+q(0))$
$=\alpha L(p)+\beta L(q)$
5:
The operator $M:\mathbb{R}^2\to\mathbb{R}^2$ given by $M(\mathbf{v})=\begin{pmatrix}v_2^2\\v_1^2\end{pmatrix}$ f0r $\mathbf{v}=\begin{pmatrix}v_1\\v_2\end{pmatrix}$ is not linear.
$M(2\mathbf{x})=\begin{pmatrix}4x_2^2\\4x_1^2\end{pmatrix}=4M(\mathbf{x})\neq 2M(\mathbf{x})$
6:
Let $A\in\mathbb{R}^{m\times n}$ and $L_A:\mathbb{R}^n\to\mathbb{R}^m$ given by $L_A(\mathbf{v})=A\mathbf{v}$ for $v\in\mathbb{R}^n$ is linear:
$L_A(\alpha\mathbf{x}+\beta\mathbf{y})=A(\alpha\mathbf{x}+\beta\mathbf{y})=\alpha A\mathbf{x}+\beta A\mathbf{y}$
7:
Let $A\in\mathbb{R}^{m\times m}$, $B\in\mathbb{R}^{n\times n}$, and $L_{A,B}:\mathbb{R}^{m\times n}\to\mathbb{R}^{m\times n}$ given by:
$L_{A,B}=AX+XB$ is linear:
$L_{A,B}(\alpha X+\beta Y)=A(\alpha X+\beta Y)+(\alpha X+\beta Y)B=\alpha(AX+XB)+\beta(AY+YB)=\alpha L_{A,B}(X)+\beta L_{A,B}(Y)$ so linear.

Theorem:

Let $L : V \to W$ linear transformation then:
(1) $L(\mathbf{0}_V) = \mathbf{0}_w$
(2)=$L(-\mathbf{v}) = -L(\mathbf{v})$ for all $v \in V$
(3) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ vectors of V and α_2, α_n are scalers then:
$L(\alpha_1 v_1 + \ldots + \alpha_n v_n) = \alpha_1 L(v_1) + \ldots + \alpha_n L(v_n)$

Proof:

(1) $L(\mathbf{0}_v) = L(0 \cdot \mathbf{v}) = 0 \cdot L(\mathbf{v}) = \mathbf{0}_w$
(2) $\mathbf{0}_w = L(\mathbf{0}_v) = L(\mathbf{v}_(- \mathbf{v})) = L(\mathbf{v}) + L(-\mathbf{v}) \Rightarrow L(-\mathbf{v}) = -L(\mathbf{v})$
(3) Repeated application of the definition.

Definition:

(1) Let $L : V \to W$ linear transformation. The kernel of L defined by:
$\ker(L) : \{v \in V | L(\mathbf{v} = \mathbf{0}_w\}$
(2) Let S be subspace V, the Image of S under L denoted by $L(S)$ defined by:
$L(S) := \{w \in W | w = L(\mathbf{v}) \text{ for some } v \in S\}$
$\text{range}(S) = L(V)$
(3) If $L : V \to W$ linear transformation and S subspace of V, then both $\ker L$ and $L(S)$ subspaces.

Proof:

3:
(a) $\ker(L)$ subspace:
(-) $\mathbf{0}_v \in \ker L \neq \emptyset$
(-) Let $\mathbf{v} \in \ker L$ and α be a scaler:
(-) $L(\alpha\mathbf{v}) = \alpha L(\mathbf{v}) = \alpha \mathbf{0}_w = \mathbf{0}_q$
$\Rightarrow \alpha\mathbf{v} \in \ker L$
(b) Let $\mathbf{v}_1, \mathbf{v}_2 \in \ker L$
(-) $L(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0}_w \Rightarrow \mathbf{v}_1 + \mathbf{v}_2 \in \ker L$
(c) $L(S)$ subspace:
(-) $\mathbf{0}_w = L(\mathbf{0}_v) \in L(S) \neq \emptyset$
(-) Let $w \in L(s)$ and α scaler:
$a\mathbf{w} = \alpha L(\mathbf{v}) = L(\alpha\mathbf{v}) \Rightarrow \alpha\mathbf{w} \in L(S)$
(-) Let $\mathbf{w}_1, \mathbf{w}_2 \in L(S)$
$\mathbf{w}_1 + \mathbf{w}_2 = L(\mathbf{v}_1 + \mathbf{v}_2) \Rightarrow \mathbf{w}_1 + \mathbf{w}_2 \in L(S)$

Example:

1:
$D_k : C^k[a,b] \to C[a,b]$ given by $D_k(f) = f^{(k)}$ and let S be supspace of $C^k[a,b]$ spanned by the functions $x \mapsto e^{\lambda x}$ for $\lambda > 0$

$\ker D_k = \{f \in C^k[a,b] | D_k(f) = \mathbf{0}_{C[a,b]}\} = \{f \in C^k[a,b] | f^{(k)} = \mathbf{0}_{C[a,b]}\} = P_k$

$D_k(P_n) = \{g \in C[a,b] | g = D_k(f) \text{ for some } f \in P_n\}$ where $n \geq k$
$D_k(P_n) = \{g \in C[a,b] | g = f^{(k)} \text{ for some } f \in P_n\} = P_{n-k}$

$D_k(S) = \{g \in C[a,b] | g = D_k(f) \text{ for some } f \in S\} = \{g \in C[a,b] | g = f^{(k)} \text{ for some } f \in S\} = S$
2:
$L_{A,B} : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ given by $L_{A,B}(X) = AX + XB$ where $A = B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\text{Ker} L_{A,B} = \{X \in \mathbb{R}^{2\times 2} | L_{A,B}(X) = \mathbf{0}_{\mathbb{R}^{2\times 2}}\} = \{X \in \mathbb{R}^{2\times 2} | AX + XB = \mathbf{0}_{\mathbb{R}^{2\times 2}}\}$
$\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} | \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$
$= \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} | \begin{bmatrix} c & d \\ a & b \end{bmatrix} + \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\} = \{\begin{bmatrix} a & b \\ -b & -a \end{bmatrix} | a, b \in \mathbb{R}\}$
$L_{A,B}(\mathbb{R}^{2\times 2}) = \{Y \in \mathbb{R}^{2\times 2} | Y = L_{A,B}(X) \text{ for some } X \in \mathbb{R}^{2\times 2}\}$
$L_{A,B}(\mathbb{R}^{2\times 2}) = \{Y \in \mathbb{R}^{2\times 2} | Y = AX + ZB \text{ for some } X \in \mathbb{R}^{2\times 2}\}$
$= \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} | \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\}$
$= \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} | \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} g & h \\ e & f \end{bmatrix} + \begin{bmatrix} f & e \\ h & g \end{bmatrix}\} = \{\begin{bmatrix} a & b \\ b & a \end{bmatrix} | a, b \in \mathbb{R}\}$

Lecture 10

Matrix representations:

(1): If $L : \mathbb{R}^n \rightarrow \mathbf{R}^m$, there exists $A_{m\times n}$ s.t. $L(\mathbf{x}) = A\mathbf{x}$

(2) $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $F = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ ordered basis for vector space $V \& W$
Let $L : V \rightarrow W$ be a lin. transformation.
$A_{m\times n}$ defined by $\mathbf{a}_j = [L(\mathbf{v}_j]_F$ for $j = 1, \ldots, n$
Then $[L(\mathbf{v})]_F = A[\mathbf{v}]_E$ for all $\mathbf{v} \in V$
The matrix A is called the matrix representation of L relative to the bases E and F

If A is the matrix representing L relative to the bases E and F and if:
$\mathbf{x} = [\mathbf{v}]_E$ and $\mathbf{y} = [\mathbf{w}]_F$
Then L maps $\mathbf{v}$ to $\mathbf{w} \Leftrightarrow A$ maps $\mathbf{x}$ to $\mathbf{y}$
We can make a schedule of this:

$$\begin{array}{ccc} \mathbf{v} \in V & \xrightarrow{L} & \mathbf{w} \in W \\ \updownarrow & & \updownarrow \\ \mathbf{x} = [\mathbf{v}]_E \in \mathbb{R}^n & \xrightarrow{A} & \mathbf{y} = [\mathbf{w}]_F \in \mathbb{R}^m \end{array}$$

Proof:

$\mathbf{v} = x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n$

$L(\mathbf{v}_j) + a_{1j}\mathbf{w}_1 + \ldots + a_{mj}\mathbf{w}_m = \sum\limits_{i=1}^{m} a_{ij}\mathbf{w}_i$

$L(v) = L(x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n) = \sum\limits_{j=1}^{n} x_j L(\mathbf{v}_j)$

$L(\mathbf{v}) = \sum\limits_{j=1}^{n} x_j (\sum\limits_{i=1}^{m} a_{ij}\mathbf{w}_i) = \sum\limits_{i=1}^{m} (\sum\limits_{j=1}^{n} a_{ij}x_j)\mathbf{w}_i$

$$[L(\mathbf{v})]_F = \begin{bmatrix} \sum\limits_{j=1}^{n} a_{1j}x_j \\ \vdots \\ \sum\limits_{j=1}^{n} a_{mj}x_j \end{bmatrix} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} A\mathbf{v}_E$$

Example:

1:
We use the following schedule:

$$\begin{array}{ccc} g & \xrightarrow{D} & h \\ \updownarrow & & \updownarrow \\ \mathbf{v} \in \mathbb{R}^2 & \xrightarrow{A} & \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \end{array}$$

$h(x) = ae^{\lambda x}\cos(\omega x) + be^{\lambda x}\sin(\omega x)$
$\int h(x)dx = g(x) \Leftrightarrow D(g) = h$
$V = \text{span}\{e^{\lambda x}\cos(\omega x), e^{\lambda x}\sin(\omega x)\}$
$D : V \to V$ given by $D(f) = f'$
$E = F = (e^{\lambda x}\cos(\omega x), e^{\lambda x}\sin(\omega x))$
$D(e^{\lambda x}\cos(\omega x)) = \lambda e^{\lambda x}\cos(\omega x) - \omega e^{\lambda x}\sin(\omega x)$
$D(e^{\lambda x}\sin(\omega x)) = \omega euler^{\lambda x}\cos(\omega x) + \lambda e^{\lambda x}\sin(\omega x)$
So we can make the following transition matrix:
$A = \left[\begin{smallmatrix} \lambda & \omega \\ -\omega & \lambda \end{smallmatrix}\right]$ and from the schedule we know $A\mathbf{v} = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ so: $\mathbf{v} = \frac{1}{\lambda^2+\omega^2}\left[\begin{smallmatrix} a\lambda - b\omega \\ a\omega + b\lambda \end{smallmatrix}\right]$
$\int ae^{\lambda x}\cos(\omega x) + be^{\lambda x}\sin(\omega x)dx = \frac{a\lambda-b\omega}{\lambda^2+\omega^2}e^{\lambda x}\cos(\omega x) + \frac{a\omega+b\lambda}{\lambda^2+\omega^2}e^{P}\lambda x\sin(\omega x)$

2:
The linear operator $L : P_4 \to P_4$ is given by $L(p) = p(0) = p(1)x + p(2)x^2 + p(3)x^3$.
Let $E = (1, x, x^2, x^3)$ and $F = 1, 1, +x, x + x^2, x^2 + x^3$. Find A

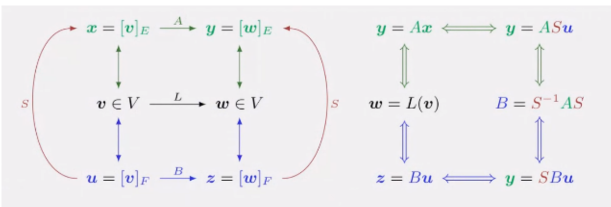
$$\begin{array}{rcccl} L(1) & = & 1\cdot 1 + 1\cdot x + 1\cdot x^2 + 1\cdot x^3 & = & 0\cdot 1 + 1\cdot(1+x) + 0\cdot(x+x^2) + 1(x^2+x^3) \\ L(x) & = & 0\cdot 1 + 1\cdot x + 2\cdot x^2 + 3x^3 & = & -2\cdot 1 + 2\cdot(1+x) - 1(x+x^2) + 3\cdot(x^2+x^3) \\ L(x^2) & = & 0\cdot 1 + 1\cdot x + 4\cdot x^2 + 9\cdot x^3 & = & -6\cdot 1 + 6\cdot(1+x) - 5\cdot(x+x^2) + 9\cdot x^2 + x^3) \\ L(x^3) & = & 0\cdot 1 + 1\cdot x + 8\cdot x^2 + 27\cdot x^3 & = & -20\cdot 1 + 20(1+x) - 19(x+x^2) + 27(x^2+x^3) \end{array}$$

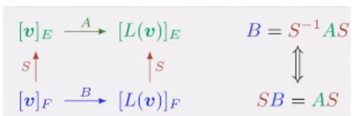
So $A = \begin{bmatrix} 0 & -2 & -6 & -20 \\ 1 & 2 & 6 & 20 \\ 0 & 1 & -5 & -19 \\ 1 & 3 & 9 & 27 \end{bmatrix}$

Matrix representations w.r.t. two bases

Let $L : V \to V$ linear operator: $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $F = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ ordered bases vector space V.
S : transition matrix, basis change F to E
If A and B are matrices representing L w.r.t E and F respectively, then $B = S^{-1}AS$

Proof:





Definition:

Let A and B be $n \times n$ matrices. We say that B is similar to A if there exists a nonsingular matrix S such that $B = S^{-1}AS$
If B similar to A then A is similar to B

Example:

Population dynamics:
$\begin{bmatrix} m_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} m_k \\ s_k \end{bmatrix}$
$\begin{bmatrix} m_n \\ s_n \end{bmatrix} = M^n \begin{bmatrix} m_0 \\ s_0 \end{bmatrix}$
$L(\mathbf{x}) = M(\mathbf{x})$
$E = (\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix})$ and $F = (\begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix})$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
Where we can make the following transition matrix:
$S = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$
We also know that:
$L(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 0.7 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$L(\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.8 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
Where we can make the following transition matrix:
$A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$

We also know that:
$L(\begin{bmatrix} 2 \\ 3 \end{bmatrix}) = 1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$L(\begin{bmatrix} -1 \\ 1 \end{bmatrix}) = 0 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 0.5 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
Where we can make the following transition matrix:
$B = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$

If we calculate $SB = AS$ we will see that it is correct.
2:
The linear operator $L : P_4 \to P_4$ is given by $L(p) = p(0) + p(1)x + p(2)x^2 + p(3)x^3$.
Let $E = (1, x, x^2, x^3)$ and $F(1, 1 + x, x + x^2, x^2 + x^3)$. Find A, B and S

$$\begin{aligned} 1 &= 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ 1 + x &= 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ x + x^2 &= 0 \cdot 1 + 1 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 \\ x^2 + x^3 &= 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 1 \cdot x^3 \end{aligned}$$

Which give us the matrix transition: $S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{aligned} L(1) &= 1 \cdot 1 + 1 \cdot x + 1 \cdot x^2 + 1 \cdot x^3 \\ L(x) &= 0 \cdot 1 + 1 \cdot x + 2 \cdot x^2 + 3 \cdot x^3 \\ L(x^2) &= 0 \cdot 1 + 1 \cdot x + 4 \cdot x^2 + 9 \cdot x^3 \\ L(x^3) &= 0 \cdot 1 + 1 \cdot x + 8 \cdot x^2 + 27 \cdot x^3 \end{aligned}$$

Which give us the matrix transition: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix}$

$$\begin{aligned} L(1) &= 1 \cdot 1 + 1 \cdot x + 1 \cdot x^2 + 1 \cdot x^3 &&= 0 \cdot 1 + 1 \cdot (1 + x) + 0 \cdot (x + x^2) + 1 \cdot (x^2 + x^3) \\ L(x) &= 0 \cdot 1 + 1 \cdot x + 2 \cdot x^2 + 3 \cdot x^3 &&= -2 \cdot 1 + 3 \cdot (1 + x) - 1 \cdot (x + x^2) + 4 \cdot (x^2 + x^3) \\ L(x^2) &= 0 \cdot 1 + 1 \cdot x + 4 \cdot x^2 + 9 \cdot x^3 &&= -8 \cdot 1 + 8 \cdot (1 + x) - 6 \cdot (x + x^2) + 12 \cdot (x^2 + x^3) \\ L(x^3) &= 0 \cdot 1 + 1 \cdot x + 8 \cdot x^2 + 27 \cdot x^3 &&= -26 \cdot 1 + 26 \cdot (1 + x) - 24 \cdot (x + x^2) + 36 \cdot (x^2 + x^3) \end{aligned}$$

Which give us the matrix transition: $B = \begin{bmatrix} 0 & -2 & -8 & -26 \\ 1 & 3 & 8 & 26 \\ 0 & 1 & -6 & -24 \\ 1 & 4 & 12 & 36 \end{bmatrix}$

Lecture 11:

Orthogonality:

Let $\mathbf{x}$ and $\mathbf{y}$ be two column vectors in $\mathbb{R}^n$. The product $\mathbf{x}^T\mathbf{y}$ is called the scaler product of $\mathbf{x}$ and $\mathbf{y}$

So $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

Then: $\mathbf{x}^T\mathbf{y} = x_1y_1 + \ldots + x_ny_n$

Eucledian length of a vector $\mathbf{x} \in \mathbb{R}^n$ defined by:
$\|\mathbf{x}\| := \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

Observation:

$||\mathbf{x}|| \geq 0$ for all $\mathbf{x}$
$||\mathbf{x}|| = 0$ iff $\mathbf{x} = 0$

In particular: $||\mathbf{x}|| = \begin{cases} \sqrt{x_1^2 + x_2^2} \text{ if } x \in \mathbb{R}^2 \\ \sqrt{x_1^2 + x_2^2 + x_3^2} \text{ if } x \in \mathbb{R}^3 \end{cases}$

The distance between two vectors, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as $\| \mathbf{x} - \mathbf{y} \|$

Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are said to be orthogonal if $\mathbf{x}^T\mathbf{y} = 0$
We write $\mathbf{x} \perp \mathbf{y}$ if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal

Phytagorean law:

If $\mathbf{x} \perp \mathbf{y}$ then:
$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x} + \mathbf{y}\|^2$
Where $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$

Proof:

$\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y})^T(\mathbf{x} + \mathbf{y}) = \mathbf{x}^T\mathbf{x} + \mathbf{y}^T\mathbf{x} + \mathbf{x}^T\mathbf{y} + \mathbf{y}^T\mathbf{y} = \mathbf{x}^T\mathbf{x} + \mathbf{y}^T\mathbf{y} = \|\mathbf{x}\|^2 + \|\mathbf{y}\|$

Angle between 2 vectors:

If $\mathbf{x}$ and $\mathbf{y}$ are 2 vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and θ is the angel between them then:
$\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$

Proof:

Law of cosines we have:
$\|\mathbf{y} - \mathbf{x}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\| - 2\|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$
Thus we get:
$2\|\mathbf{x}\|\|\mathbf{y}\| \cos\theta = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - \|\mathbf{y} - \mathbf{x}\|^2$
$= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - (\mathbf{y} - \mathbf{x})^T(\mathbf{y} - \mathbf{x})$
$= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - \mathbf{x}^T\mathbf{x} + \mathbf{y}^T\mathbf{x} + \mathbf{x}^T\mathbf{y} - \mathbf{y}^T\mathbf{y} = 2\mathbf{x}^T\mathbf{y}$
$\Rightarrow \mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$

So we see that $\cos\theta = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$
When both vectors are nonzero vectors.

Definition:

Scaler projection of $\mathbf{x}$ onto $\mathbf{y}$: $\alpha = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|}$
Vector projection of $\mathbf{x}$ onto $\mathbf{y}$: $\mathbf{p} = \alpha\mathbf{u} = \alpha\frac{1}{\|\mathbf{y}\|}\mathbf{y} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y}$

Cauchy-Schwartz inequality:

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then:
$\left|\mathbf{x}^T\mathbf{y}\right| \leq \|\mathbf{x}\|\|\mathbf{y}\|$
Observation:
$= 1 \leq \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1$, because $-1 \leq \cos\theta \leq 1$

Proof:

If $\mathbf{y} = \mathbf{0}$ then $\left|\mathbf{x}^T\mathbf{y}\right| \leq \|\mathbf{x}\|\|\mathbf{y}\|$ holds. If $\mathbf{y} \neq \mathbf{0}$, define: $\lambda = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|^2}$
Then we have:
$0 \leq \|\mathbf{x} - \lambda\mathbf{y}\|^2$
$= (\mathbf{x} - \lambda\mathbf{y})^T(\mathbf{x} - \lambda\mathbf{y})$
$= \mathbf{x}^T\mathbf{x} - \lambda\mathbf{x}^T\mathbf{y} - \lambda\mathbf{y}^T\mathbf{x} + \lambda^2\mathbf{y}^T\mathbf{y}$
$= \|\mathbf{x}\|^2 - 2\frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{x}^T\mathbf{y} + (\frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|^2})^2\mathbf{y}^T\mathbf{y}$
$= \|\mathbf{x}\|^2 - 2\frac{(\mathbf{x}^T\mathbf{y})^2}{\|\mathbf{y}\|^2} + \frac{(\mathbf{x}^T\mathbf{y})^2}{\|\mathbf{y}\|^2}$
$= \|\mathbf{x}\|^2 - \frac{(\mathbf{x}^T\mathbf{y})^2}{\|\mathbf{y}\|^2}$
This implies that $(\mathbf{x}^T\mathbf{y})^2 \leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2$
Hence we get $\left|\mathbf{x}^T\mathbf{y}\right| \leq \|\mathbf{x}\|\|\mathbf{y}\|$

Notation:

If $P_1 \& P_2$ points in 3-space, then the vector from P_1 to P_2 by $P_1\vec{}P_2$

Orthogonal subspaces:

Two subspaces X and Y of $\mathbb{R}^n$. orthogonal if $\mathbf{x}^T\mathbf{y} = 0$ for every $\mathbf{x} \in X$ and $\mathbf{y} \in Y$
We write $X \perp Y$ if X and Y orthogonal.

Orthogonal complement:

Let X be subspace of $\mathbb{R}^n$. Define $X^\perp := \{\mathbf{y} \in \mathbb{R}^n | \mathbf{x}^t\mathbf{y} = 0 \text{ for all } \mathbf{x} \in X\}$
The set $X^\perp$ called orthogonal complement of X

Observation:

(1) $X \perp Y$ then $X \cap Y\{\mathbf{0}\}$
(2) $X^\perp$ is a subspace

Proof observation:

(1)Let $\mathbf{x} \in X \cap Y$ then $\|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x} = 0$ so $\mathbf{x} = 0$
(2) $\mathbf{0} \in X^{\perp} \Rightarrow X^{\perp} \neq \emptyset$
Let $\mathbf{y} \in X^{\perp}$,and α a scaler.
Then for all $\mathbf{x} \in X$:
$\mathbf{x}^T(\alpha\mathbf{y}) = \alpha(\mathbf{x}^T\mathbf{y}) = \alpha \cdot 0 = 0 \Rightarrow \alpha\mathbf{y} \in X^{\perp}$
Let $\mathbf{y}_1, \mathbf{y}_2 \in X^{\perp}$, then for all $\mathbf{x} \in X$, we have:
$\mathbf{x}^T(\mathbf{y}_1 + \mathbf{y}_2) = \mathbf{x}^T\mathbf{y}_1 + \mathbf{x}^T\mathbf{x}_2 = 0 + 0 = 0 \Rightarrow \mathbf{y}_1 + \mathbf{y}_2 \in X^{\perp}$

Fundamental subspaces:

(-) Let $A \in \mathbb{R}^{m\times n}$. The null space of A and range of A defined by:
$N(A) := \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{0}\}$ and $R(A) = \{\mathbf{y} \in \mathbb{R}^m | \mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n\}$

Theorem:

Let $A \in \mathbb{R}m \times n$, then $N(A) = R(A^T)^{\perp}$ and $N(A^T) = R(A)^{\perp}$

Proof:

Prove that $N(A) \subseteq R(A^T)^{\perp}$
$\Rightarrow \mathbf{x} \in N(A) \Rightarrow A\mathbf{x} = \mathbf{0}$
$\Rightarrow \mathbf{x}^T A^T \mathbf{z} = 0, \forall \mathbf{z} \in \mathbb{R}^m$
$\Rightarrow \mathbf{x}^T\mathbf{y} = 0, \forall \mathbf{y} \in R(A^T)$
$\Rightarrow \mathbf{x} \in R(A^T)^{\perp} \Rightarrow N(A) \subseteq R(A^T)^{\perp}$

Prove that $R(A^T)^{\perp} \subseteq N(A)$
$x \in R(A^T)^{\perp} \Rightarrow \mathbf{x}^T\mathbf{y} = 0, \forall y \in R(A^T)$
$\Rightarrow \mathbf{x}^T A^T \mathbf{z} = 0, \forall z \in \mathbb{R}^m$
$\Rightarrow A\mathbf{x} = \mathbf{0}$
$\Rightarrow \mathbf{x} \in N(A) \Rightarrow R(A^T)^{\perp} \subseteq N(A)$

Because $R(A^T)^{\perp} \subseteq N(A)$ and $N(A) \subseteq R(A^T)^{\perp}$, we can conclude that $R(A^T)^{\perp} = N(A)$

Subspaces vs. their orthogonal complements

If S subspace of $\mathbb{R}^n$ then $\dim S + \dim S^{\perp} = n$. Moreover if $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is a basis for S and $\{\mathbf{x}_{r+1}, \ldots, \mathbf{x}_n\}$ is a basis for $S^{\perp}$, then $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ basis for $\mathbb{R}^n$

Proof:

If $S = \{\mathbf{0}\}$, then $S^{\perp} = \mathbb{R}^n$ and $\dim S + \mathrm{Dim} S^{\perp} = 0 + n = n$.
If $S \neq \{\mathbf{0}\}$, then let $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ basis for S
Define $X = [\mathbf{x}_1\ \mathbf{x}_2 \cdots x_r]$. Then $S = R(X)$ and that implies that $S^{\perp} = N(X^T)$. From rank-nullity theorem, we have:
$\dim S + \dim S^{\perp} = \dim R(x) + \dim N(X^T) = \mathrm{Rank}(X^T) + \mathrm{null}(X^T) = r + n - r = n$
In order to show $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ basis for $\mathbb{R}^n$, enough to prove these vectors linearly independent.:
$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n = \mathbf{0}$
Let $\mathbf{y} = c_1\mathbf{x}_1 + \ldots + c_r\mathbf{x}_r$ and $\mathbf{z} = c_{r+1}\mathbf{x}_{r+1} + \ldots + c_n\mathbf{x}_n$. Then we have $\mathbf{y} + \mathbf{z} = \mathbf{0}$ and hece $\mathbf{y} = -\mathbf{z}$

So $\mathbf{y}$ and $\mathbf{z}$ belongs to $S \cap S^{\perp} = \{\mathbf{0}\}$. HEnce:
$c_1\mathbf{x}_1 + \ldots + c_r\mathbf{x}_r = \mathbf{0}$
$c_{r+1}\mathbf{x}_{r+1} + \ldots + c_n\mathbf{x}_n = \mathbf{0}$
Therefore,$c_1 = \ldots = c_r = 0 = c_{r+1} = \ldots = c_n$

Orthogonal subspaces:

Let U and V subspaces of vector space W if each W can be wiitten uniquely as a sum of $\mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in U$ and $\mathbf{v} \in V$ then we say that W direct sum of U and V written as:
$W = U \bigoplus V$

Theorem:

If S subspace $\mathbb{R}^n$ then $\mathbb{R}^n = S \bigoplus S^{\perp}$

Proof:

The theorem at fundamental subspaces implies that ever $\mathbf{x} \in \mathbb{R}^n$, can be written as:
$\mathbf{x} = c_1\mathbf{x} + \ldots + c_r\mathbf{x}_r + c_{r+1}\mathbf{x}_{r+1} + \ldots + c_n\mathbf{x}_n = \mathbf{u} + \mathbf{v}$
where $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ basis for S,and $\{\mathbf{x}_{r+1}, \ldots, \mathbf{x}_n\}$ a basis for $S^{\perp}$. Let $S \ni \mathbf{u} = c_1\mathbf{x}_1 + \ldots + c_r\mathbf{x}_r$ and $\mathbf{v} = c_{r+1}\mathbf{x}_{r+1} + \ldots + c_n\mathbf{x}_n \in S^{\perp}$
For uniqueness, suppose $\mathbf{x} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w} \in S$ and $\mathbf{z} \in S^{\perp}$ then we have: $\mathbf{u} + \mathbf{v} = \mathbf{x} = \mathbf{w} + \mathbf{z} \Rightarrow S \ni \mathbf{u} - \mathbf{w} = \mathbf{z} - \mathbf{v} \in S^{\perp}$.
Since $S \cap S^{\perp} = \{\mathbf{0}\}$, we have $\mathbf{u} - \mathbf{w}$ and $\mathbf{z} = \mathbf{v}$

Theorem:

If S subspace of $\mathbb{R}^n$, then $(S^{\perp})^{\perp} = S$

proof:

$\mathbf{x} \in S \Rightarrow \mathbf{x} \perp \mathbf{y}$, for all $\mathbf{y} \in S^{\perp} \Rightarrow x \in (S^{\perp})^{\perp} \Rightarrow S \subseteq (S^{\perp})^{\perp}$
$\mathbf{x} \in (S^{\perp})^{\perp} \Rightarrow \mathbf{x} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in S$ and $\mathbf{v} \in S^{\perp}$
So $\mathbf{v} \in S^{\perp} \Rightarrow \mathbf{v} \perp \mathbf{x}$, and $\mathbf{v} \perp \mathbf{u} \Rightarrow 0 = \mathbf{v}^T\mathbf{x} = \mathbf{v}^T(\mathbf{u} + \mathbf{v}) = \mathbf{v}^T\mathbf{v} = \|\mathbf{v}\|^2$
$\Rightarrow \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{u} \in S \Rightarrow (S^{\perp})^{\perp} \subseteq S$
So therefore $S = (S^{\perp})^{\perp}$

Lecture 12

$A \in \mathbb{R}^{m\times n}$ and $m > n$ and $\mathbf{b} \in \mathbb{R}^m$.
For each $\mathbf{x} \in \mathbb{R}^n$ define residual: $r(\mathbf{x}) = A\mathbf{x} -$
We want to find $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $\|r(\hat{\mathbf{x}})\| \leq \|r(\mathbf{x}\|$ For all $x \in \mathbb{R}^n$.
So we want to minimize $\|r(\mathbf{x})\|$ equivalently $\|r\mathbf{x}\|^2$
LEAST SQUARE SOLUTION OF THE SYSTEM $A\mathbf{x} = \mathbf{b}$: A vector $\hat{\mathbf{x}}$ satisfying the inequality.

Towards a solution:

S subspace $\mathbb{R}^m$, for every $\mathbf{b} \in \mathbb{R}^m$, unique $\mathbf{p}$ of S closest to $\mathbf{b}$ so:
$\|\mathbf{b} - \mathbf{p}\| < \|\mathbf{b} - \mathbf{y}\|$ where $\mathbf{y} \neq \mathbf{p}$ in S
$\mathbf{b} \in S$ closest to $\mathbf{b}$ iff $\mathbf{b} - \mathbf{p} \in S^{\perp}$
The vector $\mathbf{p}$ is said to the projection of $\mathbf{b}$ onto S

Proof

S subspace $\mathbb{R}^m \Rightarrow \mathbb{R}^m = S \bigoplus S^{\perp}$
$\Rightarrow \mathbf{b} = \mathbf{p} + \mathbf{z}$ where $\mathbf{b} \in \mathbb{R}^m, \mathbf{p} \in S \& \mathbf{z} \in S^{\perp}$
$\|\mathbf{b} - \mathbf{y}\|^2 = \|(\mathbf{b} - \mathbf{p}) + (\mathbf{p} - \mathbf{y})\|^2$
Since $\mathbf{b} - \mathbf{p} = \mathbf{z} \in S^{\perp}$ and $\mathbf{p} - \mathbf{y} \in S$:
$\|\mathbf{b} - \mathbf{y}\|^2 = \|\mathbf{b} - \mathbf{p}\|^2 + \|\mathbf{p} - \mathbf{y}\|^2$
Due to Phytagorean law.Since $\mathbf{y} \neq \mathbf{p} \Rightarrow \|\mathbf{p} - \mathbf{y}\| > 0$
As such we can conclude that $\|\mathbf{b} - \mathbf{p}\| < \|\mathbf{b} - \mathbf{y}\|$

Solution:

Observation and theorems:

1:
OBS: If $\hat{\mathbf{x}}$ is a least square solution of the system $A\mathbf{x} = \mathbf{b}$ and $\mathbf{p} = A\hat{\mathbf{x}}$,then $\mathbf{p}$ is the vector in $R(A)$ that is the closest to $\mathbf{b}$
THM: Let S subspace of $\mathbb{R}^m$. For every $\mathbf{b} \in \mathbb{R}^m$, unique vector $\mathbf{p}$ of S closest to $\mathbf{b}$ that is $\|\mathbf{b} - \mathbf{p}\| < \|\mathbf{b} - \mathbf{y}\|$, for all $\mathbf{y} \neq \mathbf{p}$, in S. MOreover, a vector $\mathbf{p} \in S$ is closest to a given vector $\mathbf{b}$ iff $\mathbf{b} - \mathbf{p} \in S^{\perp}$ **2:**
OBS: Take $S = R(A)$
A vector $\hat{\mathbf{x}}$ is a least square solution of $A\mathbf{x} = \mathbf{b}$ iff:
$\mathbf{p} = A\hat{\mathbf{x}}$ is the vector in $R(A)$ that is the closest to $\mathbf{b}$ iff:
$\mathbf{b} - \mathbf{p} = \mathbf{b} - A\hat{\mathbf{x}} \in R(A)^{\perp} = N(A^T)$ iff $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$
Iff $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$
THM:If $A \in \mathbb{R}^{m\times n}$ is of rank n, then the normal equations:
$A^T A\mathbf{x} = A^T\mathbf{b}$ have a unique solution:
$\hat{\mathbf{x}} = (A^T A)^{-1} A^T\mathbf{b}$

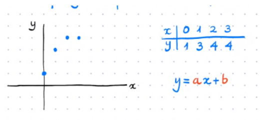
proof

Enough to prove that $A^T A$ nonsingular.
Let $\mathbf{z}$ vector s.t. $A^T A\mathbf{z} = \mathbf{0}$
Then we have $0 = \mathbf{z}^T A^T A\mathbf{z} = \|A\mathbf{z}\|^2 \Rightarrow A\mathbf{z} = \mathbf{0}$
Since $\mathrm{rank}(A) = n$ columns of A linearly independent. $\Rightarrow \mathbf{z} = \mathbf{0} \Rightarrow A^T A$ nonsingular.

Example:

1:

Find the best line fitting to the points $(0,1), (1,3), (2,4)$ and $(3,4)$



$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$

So the left matrix is matrix A and the right matrix is the b vector.

When we do: $A^T A$:

	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix}$

So $A^T A = \begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix}$

At te same way calculate $A^T b = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$

So now we want to calculate:

$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$

$\det\left(A^T A\right) = 20$

So $(A^T A)^{-1} = \frac{1}{20} \begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix}$

So $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix} \begin{pmatrix} 23 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ +\frac{3}{2} \end{pmatrix}$

So best line for $a = 1$ and $b = +\frac{3}{2}$ So $y = x + \frac{3}{2}$

Lecture 13

Eigen values and eigen vectors:

A square matrix.
Eigenvalue or characteristic value: (λ) exists $\mathbf{x}$ nonzero s.t. $A\mathbf{x} = \lambda\mathbf{x}$
Eigen vector or characteristic vector: $\mathbf{x}$

Example:

$\left[\begin{smallmatrix}4 & 2\\1 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}6\\3\end{smallmatrix}\right] = 3 \cdot \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}4 & 2\\1 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}4\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}12\\6\end{smallmatrix}\right] = 3 \cdot \left[\begin{smallmatrix}4\\2\end{smallmatrix}\right]$

observation

Following statements equivalent:
(-) λ eigenvalue of A
(-) $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
(-) $N(\lambda I - A) \neq \{\mathbf{0}\}$
(-) $(\lambda I - A)$ is singular
(-0 $\det(\lambda I - A) = 0$

The book says $A - \lambda I$ instead of $\lambda I - A$ but I am not sure or there is any difference.

Terminology:

$N(\lambda I - A)$ eigen space cooresponding to λ where λ eigen value A
characteristic polynomial: $\rho_a(\lambda) = \det(\lambda I_A)$
If $A \in \mathbb{R}^{n\times n}$, then $\rho_A(\lambda)$ polynomial of degree n

complex eigenvalues of real matrices:

If A is a square matrix, with real entries, then charecterisatic polynomial has real coefficients.
As such, all its nonreal eigenvalues occur in conjugate pairs.
Also the eigenvectors occurs in conjugate pairs:
$A\mathbf{z} = \lambda\mathbf{z} \Rightarrow A\overline{\mathbf{z}} = \overline{A}\overline{\mathbf{z}} = \overline{A\mathbf{z}} = \overline{\lambda\mathbf{z}} = \overline{\lambda}\overline{\mathbf{z}}$

Example:

without using the observation we made:
$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. The characteristic polynomial $\rho_A(\lambda)$ given by:
$\rho_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 3 & -1 \\ -1 & \lambda+2 & -1 \\ -1 & 3 & \lambda-2 \end{vmatrix} = \lambda(\lambda-1)^2$
We want htat $\det(\lambda I - A) = 0$
So $\lambda(\lambda-1)^2 = 0 \Rightarrow lambda = 0 \vee (\lambda-1)^2 = 0 \Rightarrow \lambda_1 = 0 \vee \lambda_{2,3} = 1$
Because a polynomial of degree n has n roots.
Eigen vector for $\lambda_1 = 0$:
$\mathbf{0} = (\lambda_1 I - A)\mathbf{x}$
$\begin{bmatrix} -2 & 3 & 1 \\ -1 & 2 & -1 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ This leads to: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Eigen vector for $\lambda_{2,3} = 1$:
$\mathbf{0} = (\lambda_1 I - A)\mathbf{x}$
$\begin{bmatrix} -1 & 3 & 1 \\ -1 & 3 & -1 \\ -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ This leads to: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ **2:**
$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$
The characteristic polonomial $\rho_A(\lambda)$ given by:
$\rho_A = \det(\lambda I - A) = \begin{bmatrix} \lambda-1 & -2 \\ 2 & \lambda-1 \end{bmatrix} = (\lambda-1)^2 + 4$
$\lambda_{1,2} = 1 \pm 2i$
$\lambda_1 = 1 + 2i$
$\mathbf{0} = (\lambda_1 I - A)\mathbf{x} = \begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ this leads to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ i \end{bmatrix}$
$\lambda_2 = 1 - 2i$
$\mathbf{0} = (\lambda_2 I - A)\mathbf{x} = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
This leads to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Product and sum of eigenvalues:

$$\rho_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & & \lambda - a_{nn} \end{vmatrix}$$
$= \lambda^n + \rho_{n-1}\lambda^{n-1} + \ldots + \rho_1\lambda + \rho_0$
$(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n)$
$\lambda^n - (\lambda_1 + \lambda_2 + \ldots + \lambda_n)\lambda^{n-1} + \ldots + (-1)^n\lambda_1\lambda_2\ldots\lambda_n$

$\rho_A(\lambda) = (\lambda - a_{11})(\lambda - a_{22})\ldots(\lambda - a_{nn}) + q(\lambda)$ where $\deg(q) \leq n-2$
$= \lambda^n - (a_{11} + a_{22} + \ldots + a_{nn})\lambda^{n-1} + \tilde{q}(\lambda)$ where $\deg(\tilde{q}) \leq n-2$

$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$
TRACE ($\mathrm{tr}(A)$): the sum of the diagonal elements of A.
$\rho_A(0) = \rho_0 = \det(-A) = (-1)^n\det(A) = (-1)^n\lambda_1\lambda_2\ldots\lambda_n$
USEFUL ABOUT DETERMINANT OF A $\det(A) = \lambda_1\lambda_2\ldots\lambda_n$

Similarity:

A and B both $n \times n$
$B\&A$ similar $\Rightarrow$ same characteristic polonomial and the same eigenvalues.

Proof:

S nonsingular matrix s.t. $B = S^{-1}AS$ then we have:
$\rho_B(\lambda) = \det\big(\lambda I - S^{-1}AS\big) = \det\big(S^{-1}(\lambda I - A)S\big) = \det\big(S^{-1}\big)\det(\lambda I - A)\det(S) = \det(\lambda I - A) = \rho_A(\lambda)$

Systems of linear differential equations:

Terminology:

Differential equations used to model dynamical systems in a variety of context including mechanical, electrical, hydraulic etc. systems.
System of linear differential equation is of the form $\mathbf{x}'(t) = A\mathbf{x}(t)$
Where $'$ denotes the derivative w.r.t. time variable t,$\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ vector-valued function and A and $n\times n$ matrix.

Initial value problem:

amount finding solution for $\mathbf{x}'(t) = A\mathbf{x}(t)$ and $\mathbf{x}(0) = \mathbf{x}_0$ for an $A_{n\times n}$ and a given $n-$vector $\mathbf{x}$

solution

Considetr $x(t) = e^{at}x_0$
real number a the exponential of a can be expressed by a power series of the form:
$e^a = 1 + a + \frac{1}{2!}a^2 + \frac{1}{3}a^3 + \ldots$
Similarly we define the matrix exponential by the power series:
$e^A := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots$
Thanks to uniform convergence of the above series we have:
$\frac{d}{dt}e^{tA} = \frac{d}{dt}(I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \ldots) = A + tA^2 + \frac{1}{2!}t^2A^3 + \ldots$
$\frac{d}{dt}e^{tA} = A(I + tA + \frac{1}{2!}t^2A^2 + \ldots) = Ae^{tA}$
Therefore, the solution of initial value problem above given by $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$.
Indeed we have $\mathbf{x}'(t) = (e^{tA}\mathbf{x}_0)' = Ae^{tA}\mathbf{x}_0 = A\mathbf{x}(t)$ and $\mathbf{x}(0) = e^{0\cdot A}\mathbf{x}_0 = \mathbf{x}_0$

matrix exponential of similar matrices:

If A and B both $n \times n$ matrices and similar, then $e^B = S^{-1}e^AS$

proof

We can prove that $B^k = S^{-1}A^kS$ for all $k = 1, 2, \ldots,$ by induction on k
From similarity we have: $B = S^{-1}AS$
Assume $B^l = S^{-1}A^lS$ for some $l \geq 1$
$B^{l+1} = B \cdot B^l = S^{-1}AS \cdot S^{-1}A^lS = S^{-1}AA^lS = S^{-1}A^{l+1}S$
Then: $e^B = I + B + \frac{1}{2!}B^2 + \ldots = I + S^{-1}AS + \frac{1}{2!}S^{-1}A^2S + \ldots = S^{-1}(I + A + \frac{1}{2!}A^2 + \ldots)S = S^{-1}e^AS$

Observation:

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & & & \\ & d_2^k & & \\ & & \ddots & \\ & & & d_n^k \end{bmatrix}$$

$$\Rightarrow e^D = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} + \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} d_1^2 & & & \\ & d_2^2 & & \\ & & \ddots & \\ & & & d_n^2 \end{bmatrix} + \dots$$

$$e^D = \begin{bmatrix} e^{d_1} & & & \\ & e^{d_2} & & \\ & & \ddots & \\ & & & e^{d_n} \end{bmatrix}$$

If a given square matrix A is similar to a diagonal matrix D, then $A = S^{-1}DS$ for some nonsingular matrix S and $e^A = S^{-1}e^D S$

Example:

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Does there exists a nonsingular $S \in \mathbb{R}^{2\times 2}$ s.t. $A = S^{-1}DS$
So $SA = DS$
$D = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$ and $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
$SA = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$
$DS = \begin{bmatrix} ea & eb \\ fc & fd \end{bmatrix}$
So then $ea = 0$ and $fc = 0$
Suppose that $a = 0$ then $c \neq 0$ because S is a nonzero matrix. Thus $f = 0$ since $c = fd$ then $c = 0$ so contradiction so $a \neq 0$

Suppose that $e = 0$ since $a = eb$ then a is zero, which is not possible.

Lecture 14

Diagonalization:

DIAGONALIZABLE MATRIX A exists nonsingular matrix X and diagonal matrix D s.t. $X^{-1}AX = D$
We say X diagonalizes A iff $X^{-1}AX = D \Leftrightarrow AX = XD$

Example:

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $D = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$
We see $AX = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$ and $XD = \begin{bmatrix} ea & fb \\ ec & fd \end{bmatrix}$
Claim $AX = XD \Rightarrow c = d = 0$
Suppose $c \neq 0 \Rightarrow c = ea = 0 \Rightarrow e \neq 0$ contradicts $\Rightarrow ec = 0$
Suppose $d \neq 0$ since $d = fb \Rightarrow f \neq 0$ contradicts $fd = 0$ If $c = d = 0$, then X must be singular.

Definition:

An $n \times n$ matrix is diagonalizable iff it has n linearly independent eigenvectors.

proof:

SPS $A \in \mathbb{R}$ linearly independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ Then $X = [\, \mathbf{x}_1 \; \mathbf{x}_2 \; \cdots \; \mathbf{x}_n \,]$
Ntoe that $AX = [\, A\mathbf{x}_1 \; A\mathbf{x}_2 \; \ldots \; A\mathbf{x}_n \,] = [\, \lambda_1\mathbf{x}_1 \; \lambda_2\mathbf{x}_2 \; \ldots \; \lambda_n\mathbf{x}_n \,]$

$AX = [\, \mathbf{x}_1 \; \mathbf{x}_2 \; \cdots \; \mathbf{x}_n \,] = XD$ and therefore A is diagonalizable.

A diagonalizable, then $AX = XD$ for some nonsingular X and diagonal D
Hence $A\mathbf{x}_1 = d_{ii}\mathbf{x}_i$. So $\mathbf{x}_i$ eigen vector corresponding to d_{ii} nonsingularity of X implies that A has n linearly independent eigenvectors

Observation:

A diagonalizable, then $X^{-1}AX = D$ nonsingular matrix X and a diagonal matrix D,then:
Column vectors X eigenvectors of A
Diagonal elements D eigenvalues of A
X and D are not unique.
(recording columns of X and D would lead to a different pairs (X', D'))

Theorem:

If $\lambda_1, \ldots, \lambda_k$ distinct eigenvalues($\lambda_i \neq \lambda_j$ for $i \neq j$) of A with the corresponding eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$, then those vectors are linearly independent.

Proof:

r dimension subspace spanned by $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$
Suppose $r < k$
Without lost of generality, we can assume that $\mathbf{x}_1, \ldots, \mathbf{x}_r$ linearly independent. Since $\mathbf{x}_1, \ldots, \mathbf{x}_{r+1}$
SO exists scalers $c_1, \ldots, c_r$ not all zero s.t.:
$c_1\mathbf{x}_1 + \ldots + c_{r+1}\mathbf{x}_{r+1} = \mathbf{0}$

Since $\mathbf{x}_1, \ldots, \mathbf{x}_r$ linearly independent c_{r+1} must be nonzero.
Hence $c_{r+1}\mathbf{x}_{r+1} \neq 0$
and thus $c_1, c_2, \ldots, c_r$ can not be all zero.
$c_1 A\mathbf{x}_1 + \ldots + c_{r+1} A\mathbf{x}_{r+1} = \mathbf{0}$
$c_1 \lambda_1 \mathbf{x}_1 + \ldots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = \mathbf{0}$
$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \ldots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = \mathbf{0}$
This contradicts the independence of $\mathbf{x}_1, \ldots, \mathbf{x}_r$
Thus, r must equal k

distinct eigenvalues:

Any square matrix with distinct eigenvalues is diagonalizable.

Example:

$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $\rho_{A_1}(\lambda) = \lambda^2$ not diagonalizable.
$A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \rho_{A_2}(\lambda) = \lambda^2$
$\Rightarrow A_2$ already diagonal.
$\Rightarrow A_2$ diagonalizable.
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
What is e^A
$\rho_A(\lambda) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} \Rightarrow \lambda_1 = -1$ and $\lambda_2 = 1$
Eigen vectors for $\lambda_1 = -1$
$\mathbf{0} = (\lambda_1 I - A)\mathbf{x}$
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Same way eigenvectors for λ_2
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$X = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
$AX = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = XD$
$X^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
$e^A = X \cdot e^D \cdot X^{-1} = X \begin{bmatrix} e^{-1} & 0 \\ 0 & e \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e+e^{-1} & e-e^{-1} \\ e-e^{-1} & e+e^{-1} \end{bmatrix} = \begin{bmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{bmatrix}$

Markov Chains:

STOCHASTIC PROCESS: sequence experiments, which the outcome at any stage depends on chance.
MARKOV CHAIN: Stochastic process with:
(1) Set of possible outcomes or states finite.
(2) probaibility of next outcome depends only previous outcome.
(3) probabilities constant over time.

If a Markov chain with an $n \times n$ transition matrix A converges to a steady-state vector $\mathbf{x}$,then:
(1) $\mathbf{x}$ probability vector.
(2) $\lambda_1 = 1$ eigenvalue of A and $\mathbf{x}$ eigenvector beloning to λ_1

If λ_1 dominant eigenvalue stochastic matrix A, Markov chain transition A converge to steady state vector.